Chapter Two
Lissajous' Figures

2.1: Composition of Two Simple Harmonic Motion in a Straight Line:

Analytical method;- Let the two simple harmonic vibrations be represented by the equations;

$$y_1 = a_1 \sin(wt + \alpha_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y_2 = a_2 \sin(wt + \alpha_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where y_1 & y_2 are the displacement of a particle due to the two vibrations,
a_1 & a_2 are the amplitude of the two vibrations, and
α_1 & α_2 are the epoch angle of the two vibrations.

Here, the two vibrations are assumed to be of the same frequency and hence w is the same for both. The resultant displacement y of the particle is given by;

$$y = y_1 + y_2$$

$$\begin{aligned}
y &= a_1 \sin(wt + \alpha_1) + a_2 \sin(wt + \alpha_2) \\
&= a_1 \left(\sin wt \cos \alpha_1 + \cos wt \sin \alpha_1\right) + a_2 \left(\sin wt \cos \alpha_2 + \cos wt \sin \alpha_2\right) \\
&= a_1 \sin wt \cos \alpha_1 + a_1 \cos wt \sin \alpha_1 + a_2 \sin wt \cos \alpha_2 + a_2 \cos wt \sin \alpha_2 \\
&= (a_1 \sin wt \cos \alpha_1 + a_2 \sin wt \cos \alpha_2) + (a_1 \cos wt \sin \alpha_1 + a_2 \cos wt \sin \alpha_2) \\
&= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin wt + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos wt \quad \ldots\ldots\ldots\ldots(3)
\end{aligned}$$

Since the amplitudes a_1 & a_2 and the angle α_1 & α_2 are constant, The coefficients of sinwt & coswt in equation (3) can be substituted by $A\cos\varphi$ & $A\sin\varphi$

$A\cos\varphi = a_1\cos\alpha_1 + a_2\cos\alpha_2$ .... (4)
$A\sin\varphi = a_1\sin\alpha_1 + a_2\sin\alpha_2$ .... (5)

Squaring;

$A^2\cos^2\varphi = a_1^2\cos^2\alpha_1 + a_2^2\cos^2\alpha_2 + 2a_1a_2\cos\alpha_1\cos\alpha_2$ .... (4-a)
$A^2\sin^2\varphi = a_1^2\sin^2\alpha_1 + a_2^2\sin^2\alpha_2 + 2a_1a_2\sin\alpha_1\sin\alpha_2$ .... (5-a)

Adding

$$A^2\cos^2\varphi + A^2\sin^2\varphi = a_1^2\cos^2\alpha_1 + a_2^2\cos^2\alpha_2 + 2a_1a_2\cos\alpha_1\cos\alpha_2 + a_1^2\sin^2\alpha_1 + a_2^2\sin^2\alpha_2 + 2a_1a_2\sin\alpha_1\sin\alpha_2$$

$$A^2\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} = a_1^2\underbrace{(\cos^2\alpha_1 + \sin^2\alpha_1)}_{1} + a_2^2\underbrace{(\cos^2\alpha_2 + \sin^2\alpha_2)}_{1} + 2a_1a_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) \quad .... (6)$$

But, $\cos(\alpha_1 - \alpha_2) = (\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$

Eq.(6) becomes;

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2) \quad .... (6)$$

Dividing eq. (5) by (4)

$$\frac{A\sin\varphi}{A\cos\varphi} = \frac{a_1\sin\alpha_1 + a_2\sin\alpha_2}{a_1\cos\alpha_1 + a_2\cos\alpha_2} \qquad \tan\varphi = \frac{a_1\sin\alpha_1 + a_2\sin\alpha_2}{a_1\cos\alpha_1 + a_2\cos\alpha_2} \quad ....(7)$$

Equations (6) & (7) give the value of a_1, a_2 and α_1, α_2

$$y_1 = A\cos\varphi\sin wt + A\sin\varphi\cos wt$$

$$y = A\sin(wt + \varphi) \quad ....(8)$$

Equation (8) is similar to the original equation (1) & (2).
The amplitude of the resultant vibration is A and epoch angle is φ. The value is given by eq.(6),(7).
The time period of the resultant vibration is the same as the original vibrations. Thus, the resultant of two simple harmonic vibrations of the same period and acting in the same line is also a simple harmonic vibration with a resultant amplitude A and epoch angle φ.

Special case; If $\alpha_1 = \alpha_2 = \alpha$, then $A = a_1 + a_2$ and $\varphi = \alpha$

$$y = (a_1 + a_2)\sin(wt + \alpha)$$

Example:

Two SHMs acting simultaneously on a particle are given by;

$y_1 = \sin(wt + \frac{\pi}{3})$ $y_2 = 2\sin wt$

Find the equation of the resultant vibration.

Here $y_1 = \sin(wt + \frac{\pi}{3})$ and $y_2 = 2\sin wt$ Similar to

$y_1 = a_1 \sin(wt + \alpha_1)$ and $y_2 = a_2 \sin(wt + \alpha_2)$

$\therefore$ $a_1=1$, $a_2=2$, $\alpha_1=\pi/3$ and $\alpha_2=0$

The result vibration is given by; $y = A\sin(wt + \varphi)$

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2)}$$

$$A = \sqrt{1 + 4 + 2.1.2.\cos(\frac{\pi}{3} - 0)} = \sqrt{5 + 4\cos\frac{\pi}{3}} = 2.6457\, \textit{unit of length}$$

2.2: Composition of Two Simple Harmonic Vibrations of Equal Time Periods Acting at Right Angle:-

Let $x = a\sin(wt + \alpha)$.......(1)

$y = b\sin wt$...............(2)

Represent the displacements of a particle along the X- and Y-axes due to the influence of two SHVs acting simultaneously on a particle in perpendicular directions.

Here, the two vibrations are of the same time period but are of different amplitude and different phase angles.

From eq.(2); $\sin wt = \frac{y}{b}$

But $\cos^2 wt + \sin^2 wt = 1$ $\therefore \cos^2 wt = 1 - \sin^2 wt$

$\cos^2 wt = 1 - \frac{y^2}{b^2}$ $\cos wt = \sqrt{1 - \frac{y^2}{b^2}}$

From eq.(1);

$\frac{x}{a} = \sin wt \cos\alpha + \cos wt \sin\alpha$......(3)

Substituting the values of ***sinwt*** and ***coswt*** in eq.3;

$$\frac{x}{a} = \frac{y}{b}\cos\alpha + \sqrt{1-\frac{y^2}{b^2}}\sin\alpha \qquad \frac{x}{a} - \frac{y}{b}\cos\alpha = \sqrt{1-\frac{y^2}{b^2}}\sin\alpha$$

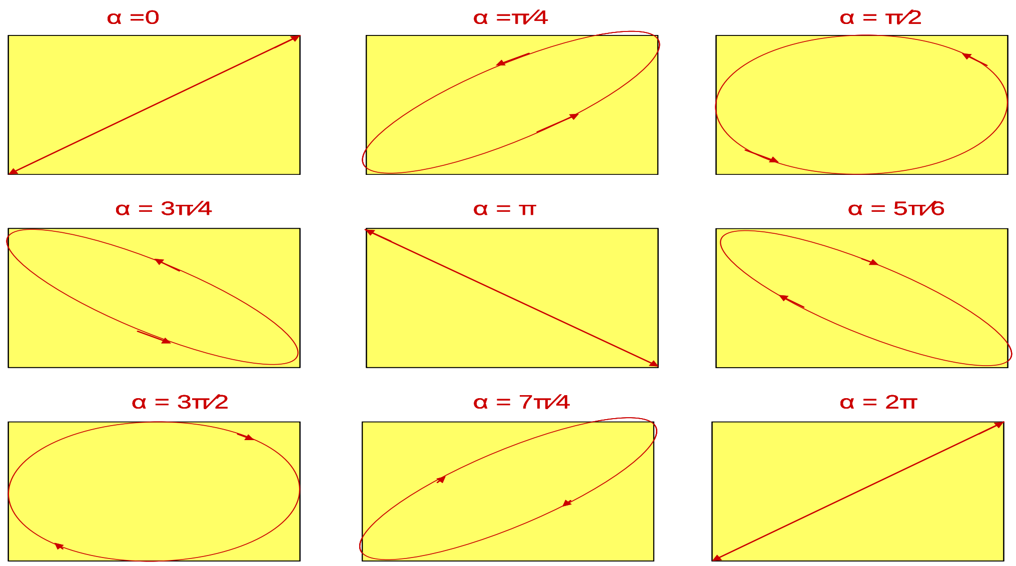
Squaring both sides;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = (1-\frac{y^2}{b^2})\sin^2\alpha \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha - \frac{y^2}{b^2}\sin^2\alpha$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}(\cos^2\alpha + \sin^2\alpha) - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha \text{.................}(4)$$

This represents the general equation of an ellipse.
Thus, due to the superimpositions of two simple harmonic vibrations, the displacement of the particle will be along a care (Fig*) given by equation (4).



The resultant vibration of the particle will depend upon the value of α
The figure represents the resultant vibration for values of α changing from 0 to 2π.

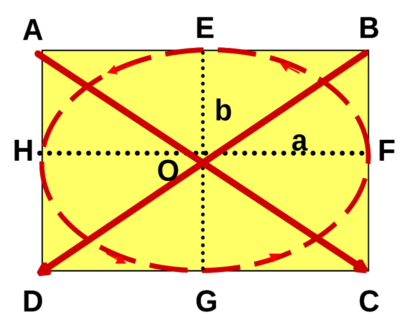
Special Case:

I) If α=0 or 2π

Cosα=1 , sinα=0

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

$$\frac{x}{a} - \frac{y}{b} = 0 \qquad \frac{x}{a} = \frac{y}{b} \rightarrow y = \frac{b}{a}x$$



II) If $\alpha=\pi$

Cosα= -1 , sinα=0

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}.1=0$$

$$\frac{x}{a}+\frac{y}{b}=0$$

$$\frac{x}{a}=-\frac{y}{b}\rightarrow y=-\frac{b}{a}x$$

This equation represents the equation of the straight line AC

III) If $\alpha=\pi/2$ or $3\pi/2$

Cosα=0 , sinα=1

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

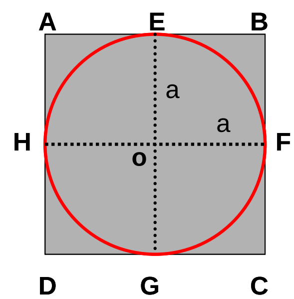
This equation represents the equation of the ellipse EHGF, with ***a*** and ***b*** as the semi-major and semi-minor axes.

IV) If $\alpha=\pi/2$ or $3\pi/2$ and a=b

Cosα=0, sinα=1

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

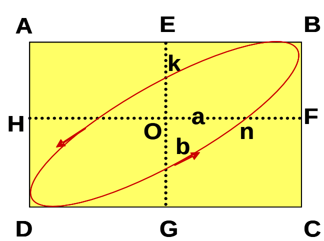
$$\frac{x^2+y^2}{a^2}=1$$

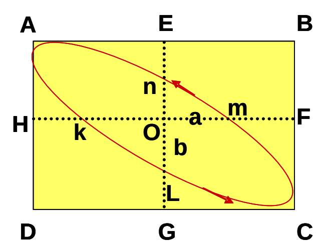


$$x^2+y^2=a^2$$

This equation represents the equation a circle.

V) If $\alpha=\pi/4$ or $7\pi/4$ the resultant vibration is an oblique ellipse KLMN as shown in figure





On the other hand if $\alpha=3\pi/4$ or $5\pi/4$ the resultant vibration is again an oblique ellipse KLMN as shown in Fig.

The cycle of changes is repeated after every time period.

2.3: Composition of Two Simple Harmonic Motion at Right Angle to Each Other and Having Time Period in the ratio1:2

Let $x = a\ \sin(2wt+\alpha)$.......(1) and

$y = b\ \sin wt$.................(2)

Here a is the amplitude for the motion along the x-axes.

B is the amplitude for the motion along the y-axes.

The phase difference between the two vibration is α.

From equation (2);

$$\frac{y}{b} = \sin wt \qquad \cos wt = \sqrt{1-\sin^2 wt}$$

From equation1;

$$\frac{x}{a} = \sin(2wt+\alpha) \qquad \frac{x}{a} = \sin 2wt\cos\alpha + \cos 2wt \sin\alpha$$

$$= 2\sin wt\cos wt\cos\alpha + \left(1-2\sin^2 wt\right)\sin\alpha$$

Substituting the values of sinwt and coswt ,

$$\frac{x}{a} = 2.\frac{y}{b}.\sqrt{1-\frac{y^2}{b^2}}\cos\alpha + \left(1-2\frac{y^2}{b^2}\right)\sin\alpha$$

$$\left[\frac{x}{a} - \left(1-\frac{2y^2}{b^2}\right)\sin\alpha\right] = 2\frac{y}{b}\cos\alpha\sqrt{1-\frac{y^2}{b^2}}$$

$$\left[\left(\frac{x}{a}-\sin\alpha\right)+2\frac{y^2}{b^2}\sin\alpha\right] = \frac{2y}{b}\cos\alpha\sqrt{1-\frac{y^2}{b^2}}$$

Squaring both sides;

$$\left(\frac{x}{a}-\sin\alpha\right)^2 + \frac{4y^4}{b^4}\sin^2\alpha + 2\left(\frac{x}{a}-\sin\alpha\right)\frac{2y^2}{b^2}\sin\alpha = \frac{4y^2\cos^2\alpha}{b^2}\left(1-\frac{y^2}{b^2}\right)$$

$$\left(\frac{x}{a}-\sin\alpha\right)^2 + \frac{4y^4}{b^4}\sin^2\alpha + 4\frac{x}{a}\frac{y^2}{b^2}\sin\alpha - \frac{4y^2\sin^2\alpha}{b^2} - 4\frac{y^2}{b^2}\cos^2\alpha + \frac{4y^4}{b^4}\cos^2\alpha$$

$$\left(\frac{x}{a}-\sin\alpha\right)^2 + \frac{4y^4}{b^4}\left(\sin^2\alpha+\cos^2\alpha\right) - \frac{4y^2}{b^2}\left(\sin^2\alpha+\cos^2\alpha\right) + \frac{4y^2}{b^2}.\frac{x}{a}\sin\alpha = 0$$

$$\left(\frac{x}{a}-\sin\alpha\right)^2+\frac{4y^4}{b^4}-\frac{4y^2}{b^2}+\frac{4y^2}{b^2}\cdot\frac{x}{a}\sin\alpha=0$$

$$\left(\frac{x}{a}-\sin\alpha\right)^2+\frac{4y^2}{b^2}\left[\frac{y^2}{b^2}+\frac{x}{a}\sin\alpha-1\right]=0\text{.........................(3)}$$

Equation(3) represents the general equation of a curve having two loops.

Special case;

(i) When $\alpha=0,\pi,2\pi,$ etc $\sin\alpha=0$

From eq.(3)

$$\frac{x^2}{a^2}+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right)=0$$

This equation represents the figure of eight and has two loops.

(ii) When $\alpha=\pi/2$ $\sin\alpha=+1$

from equation(3),

$$\left(\frac{x}{a}-1\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right)=0$$

$$\left(\frac{x}{a}-1\right)^2+\frac{4y^2}{b^2}\left(\frac{x}{a}-1\right)+\frac{4y^4}{b^4}=0$$

$$\left[\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}\right]^2=0$$

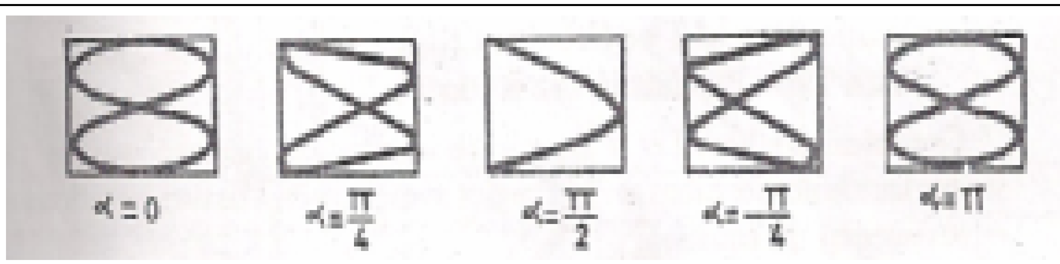
$$\left(\frac{x}{a}-1\right)+2\frac{y^2}{b^2}=0$$

$$\frac{2y^2}{b^2}=-\left(\frac{x}{a}-1\right)$$

$$y^2=-\frac{b^2}{2}\left(\frac{x}{a}-1\right)$$

$$y^2=-\frac{b^2}{2a}(x-a)$$

The represents the equation of a parapola, with vertex at (a,0).



End Chapter Two