Chapter Three

Free, Forced & Resonant Vibrations

3.1: Free Vibrations:

When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion.

Time period (T) depends only on the length (l) and the acceleration due gravity (g).

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The pendulum will continue to oscillate with the same time period and amplitude for nay length of time.

In such case, there is no loss of energy by fraction or otherwise.

In all similar case, the vibrations will be undamped free vibration. The amplitude of swing remains constant.

3.2: Undamped Vibrations:

For S.H.M., the kinetic energy for displacement y, is given by; $\frac{1}{2}m(\frac{dy}{dt})^2$ and the potential energy is $\frac{1}{2}k\,y^2$.

Where **k** is the restoring force per unit displacement, (**F=kx**).

The ***total energy*** at any instant, $= \frac{1}{2}m(\frac{dy}{dt})^2 + \frac{1}{2}k\,y^2$

For an undamped harmonic oscillator this total energy remains constant.

$$\frac{1}{2}m(\frac{dy}{dt})^2+\frac{1}{2}k\,y^2=cons\tan t \quad ----- (1)$$

Differential equation (1) with respect to time;

$$m(\frac{d^2y}{dt^2})+k\,y=0 \quad ----- (2)$$

$$\frac{d^2y}{dt^2}+\frac{k}{m}y=0 \quad ----- (3)$$ is similar to $$\frac{d^2y}{dt^2}+w^2\,y=0 \quad ----- (4)$$

Here $w^2=\frac{k}{m}$

The solution for equation (4) is; $y=a\sin(wt-\alpha)$

$$y=a\sin(\sqrt{\frac{k}{m}}\,t-\alpha)$$

The frequency of oscillation is

$$n=\frac{w}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$ and Time period is $$\therefore T=\frac{1}{n}=2\pi\sqrt{\frac{m}{k}}$$

Thus, in the case of undamped free vibrations, the differential equation is;

$$\frac{d^2y}{dt^2}+\frac{k}{m}y=0 \quad ----- (5)$$

This is only on ideal case.

3.3: Damped Vibrations:

When the pendulum vibrates in air medium, there are fictional forces and consequently is dissipated in each vibration.

The amplitude of swing decreases continuously with time and finally the oscillations die out.

Such vibration are called "free damped" vibrations.

Let $\mu \frac{dy}{dt}$ be the dissipated force due to fraction.

Therefore, the differential equation in the case of free-damped vibrations is;

$$m\frac{d^2y}{dt^2}+K\,y+\mu\frac{dy}{dt}=0 \quad ------(6) \qquad\qquad \frac{d^2y}{dt^2}+\frac{\mu}{m}\frac{dy}{dt}+\frac{K}{m}y=0 \quad ------(7)$$

This equation is similar to a general differential equation; $\frac{d^2y}{dt^2}+2b\frac{dy}{dt}+w_o^{\,2}\,y=0 \quad ------(8)$

The solution of this equation is; $y=a\,e^{-bt}\sin(wt-\alpha) \quad ------(9)$

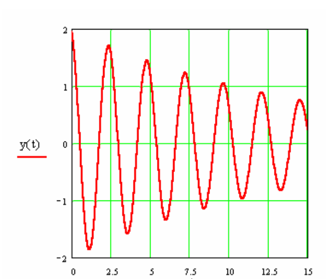
The general solution of equation (7) is also given by;

$$y=Ae^{(-b+\sqrt{b^2-w_o^{\,2}}\,)t}+(-B)\,e^{(-b-\sqrt{b^2-w_o^{\,2}}\,)t} \quad ------(10)$$

Here $2b=\frac{\mu}{m}\rightarrow\therefore b=\frac{\mu}{2m}$ and; $w_o^{\,2}=\frac{K}{m}$

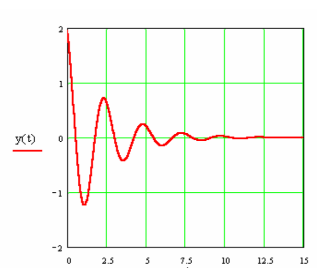
And let; $w=\sqrt{w_o^{\,2}-b^2}$ ♠; $w=\sqrt{\frac{K}{m}-\frac{\mu^2}{4m^2}}$

$$n=\frac{w}{2\pi}=\frac{1}{2\pi}\sqrt{w_o^{\,2}-b^2}$$



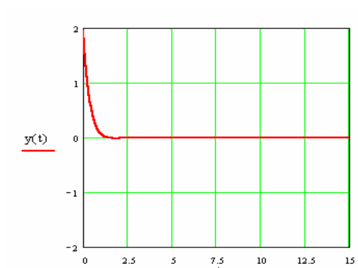
Light Damping

(large b/m)



Heavy Damping

(small b/m)



Critical Damping

$$\frac{k}{m}=\frac{\mu^2}{4m^2}$$

$$\omega'=\sqrt{\left(\frac{k}{m}-\frac{\mu^2}{4m^2}\right)}=\sqrt{\left(\omega_0^{\,2}-\frac{\mu^2}{4m^2}\right)}$$

3.4: Forced Vibration:

The time period (T) of a body executing simple harmonic motion depends on the

1- Dimensions of the body.

2- Its elastic properties.

The vibrations of such a body die out with time due to dissipation of energy.

If some external period force is constantly applied on the body, the body continues to oscillate under the influence of such external forces; such vibrations of the body are called ***forced vibrations.***

* Initially, the amplitude of the swing increases, then decrease with time, becomes a minimum and again increases.

This will be repeated if the external periodic force is constantly applied on the system.

-* The frequency of the forced vibration is different from the natural frequency of vibration of the body.

-* The ***amplitude*** of the body depends on the differences between the ***natural frequency*** and the ***frequency*** of the applied force. The amplitude will be large if difference in frequency is small and vice versa.

For forced vibration, equation (6), $m\frac{d^2y}{dt^2}+K\,y+\mu\frac{dy}{dt}=0 \quad -----(6)$ is modified in the form,

$$m\frac{d^2y}{dt^2}+Ky+\mu\frac{dy}{dt}=F\sin pt \quad -----(11)$$

Here ***p*** is the angular frequency of the applied periodic force.

The particular solution of equation (11) representing the forced vibrations is;

$$y=a\sin(pt-\alpha) \quad -----(12)$$

$$\frac{dy}{dt}=a\,p\cos(pt-\alpha) \quad -----(13)$$

$$\frac{d^2y}{dt^2}=-a\,p^2\sin(pt-\alpha)=-p^2y \quad -----(14)$$

Substituting these values in equation (11), $m\frac{d^2y}{dt^2}+Ky+\mu\frac{dy}{dt}=F\sin pt \quad -----(11)$

$$-ma\,p^2\sin(pt-\alpha)+Ka\sin(pt-\alpha)+\mu a\,p\cos(pt-\alpha)=F\sin pt$$

$$-ma\,p^2[\sin pt\cos\alpha-\cos pt\sin\alpha]+Ka[\sin pt\cos\alpha-\cos pt\sin\alpha]$$

$$+\mu a\,p[\cos pt\cos\alpha+\sin pt\sin\alpha]-F\sin pt=0 \quad ------(15)$$

When ***sin pt =1 ; cos pt = 0***

$$-ma\,p^2\cos\alpha+Ka\cos\alpha+\mu a\,p\sin\alpha-F=0 \quad ------(16)$$

When ***sin pt = 0 ; cos pt =1***

$$ma\,p^2\sin\alpha-Ka\sin\alpha+\mu a\,p\cos\alpha=0 \quad ------(17)$$

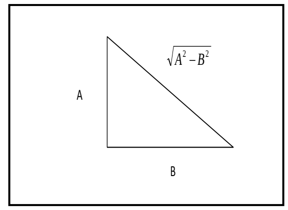
Dividing equation (17) by ***cos α*** and simplifying;

$$ma\,p^2\frac{\sin\alpha}{\cos\alpha}-Ka\frac{\sin\alpha}{\cos\alpha}+\mu a\,p\frac{\cancel{\cos\alpha}}{\cancel{\cos\alpha}}=0$$

$$ma\,p^2\tan\alpha-Ka\tan\alpha+\mu a\,p=0$$

$$\tan\alpha(ma\,p^2-Ka)+\mu a\,p=0 \qquad \cancel{a}\tan\alpha(m\,p^2-K)+\mu\cancel{a}p=0$$

$$\tan\alpha = \frac{\mu\ p}{(K - m\,p^2)} = \frac{A}{B} \quad ------ (18)$$



From equation (18)

$$\tan\alpha = \frac{A}{B} ------ (18) \qquad \sin\alpha = \frac{A}{\sqrt{A^2 - B^2}} \quad ------ (19) \qquad \cos\alpha = \frac{B}{\sqrt{A^2 - B^2}} \quad ----- (20)$$

Dividing equation (16) by ***cosα***

$$-ma\,p^2\frac{\cos\alpha}{\cos\alpha} + Ka\frac{\cos\alpha}{\cos\alpha} + \mu a\,p\frac{\sin\alpha}{\cos\alpha} - \frac{F}{\cos\alpha} = 0 \qquad -ma\,p^2 + Ka + \mu a\,p\,\tan\alpha = \frac{F}{\cos\alpha}$$

$$a(K - m\,p^2) + \mu\,a\,p\,\tan\alpha = \frac{F}{\cos\alpha} \qquad a[K - mp^2 + \mu\ p\ \tan\alpha] = \frac{F}{\cos\alpha}$$

$$\textit{But}\ K - m\,p^2 = B \quad \textit{and} \quad \mu\ p = A$$

$$a[B + A\frac{A}{B}] = \frac{F}{\dfrac{B}{\sqrt{A^2 - B^2}}} \qquad a[B + \frac{A^2}{B}] = \frac{F\sqrt{A^2 - B^2}}{B}$$

$$a[\frac{B^2 + A^2}{B}] = \frac{F\sqrt{A^2 - B^2}}{B} \qquad a[\frac{A^2 + B^2}{B}] = \frac{F\sqrt{A^2 - B^2}}{B}$$

$$a[\frac{\sqrt{A^2 + B^2}\sqrt{A^2 - B^2}}{B}] = \frac{F\sqrt{A^2 - B^2}}{B} \qquad a = \frac{F}{\sqrt{A^2 - B^2}}$$

Substituting the values of A and B, $K - m\,p^2 = B \ , \ \mu\ p = A$

$$a = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad ------- (21)$$

$$y = a\sin(pt - \alpha)$$ Or

$$y = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha) \quad ------ (22)$$

Applying the boundary condition, another solution is obtained when F=0.

This corresponds to free vibrations. In the case free vibrations the solution is;

$$y = a e^{-bt} \sin(wt - \alpha) \quad ------ (23)$$

The general solution will include both the particular solutions for free and forced vibrations.

$$\therefore y = a e^{-bt} \sin(wt - \alpha) + \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha)$$

Here $b = \frac{\mu}{2m}$

3.5: Resonance and Sharpness of Resonance:

In the case of forced vibrations, the general solution for the displacement at any instant is given by;

$$y = a e^{-bt} \sin(wt - \alpha) + \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha)$$

If the effect of viscosity of the medium is small, the amplitude $\frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}}$

Under the action of the driving force is a maximum when the denominator is a minimum. This is possible if $K - mp^2 = 0 \rightarrow K = mp^2 \ or \ p = \sqrt{\frac{K}{m}}$

Further, the amplitude will be infinite if μ is also *zero*. The oscillations will have maximum amplitude and this state of vibration of a system is called ***resonance***.

If means that, when the ***forced frequency*** is ***equal*** to the ***natural frequency*** of vibration of the body, ***resonance*** takes place.

If *friction* is present, the amplitude at resonance $= \frac{F}{\mu p} = \frac{F}{\mu\sqrt{K/m}}$

Or amplitude at resonance = $= \frac{F}{\mu}\sqrt{\frac{m}{K}}$

In the case of sound, the study of sharpness of resonance is of great importance. Sharpness of resonance refers to the fall in amplitude with change in frequency on each side of the maximum amplitude.

The particular solution for displacement in the case of forced vibrations is;

$$y = \frac{F}{\sqrt{\mu^2 p^2 - (K + mp^2)^2}} \sin(pt - \alpha) \quad ----(1)$$

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \cos(pt - \alpha) \quad ----(2)$$

The velocity (dy/dt) is maximum when cos(pt-α) is maximum, i.e., the instant at which the particle crosses the mean position.

$$\left(\frac{dy}{dt}\right)_{max} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad ----(3)$$

Kinetic energy of the vibrating particle at the instant of crossing the mean position is given by;

$$k.E = \frac{1}{2} m (\frac{dy}{dt})^2{}_{\max} = \frac{\frac{1}{2} m F^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \quad ----(4)$$

The *mean square* of the driving *force per unit mass*

$$= \frac{[\frac{0+F^2}{2}]}{m} = \frac{F^2}{2m} \quad ------(5)$$

Driving equation (4) by $\frac{F^2}{2m}$ we get kinetic energy per unit force which is called the response R.

$$R = \frac{\frac{1}{2} m F^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \div \frac{F^2}{2m} \qquad R = \frac{m^2 p^2}{\mu^2 p^2 + (K - mp^2)^2}$$

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2} + (\frac{K}{m} - p^2)^2} \quad ----(6)$$

The *natural frequency* of the system in the absence of damping is $\sqrt{\frac{K}{m}}$.

Therefore, the term $\frac{K}{m} - p^2$ in equation (6) represents the extent to which the nature frequency of the system deviates from the forced frequency.

When $\frac{K}{m} = p^2$

The nature frequency coincides with the forced frequency and the value of R will be a maximum. From equation (6)

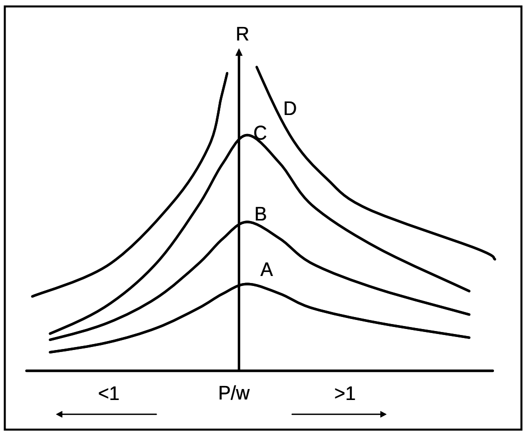
$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2}} = \frac{m^2}{\mu^2} = (\frac{m}{\mu})^2$$

The response $R \alpha \frac{1}{\mu}$

If means that the response R is inversely proportional to the fractional force. In the absence of friction, the response is a maximum.

The term $\frac{K}{m} - p^2$ in equation (6), refers to mistuning. The larger is its value, the greater is the system away from resonance.

The graph between p/w along the x-axis and the response R along the y-axis is shown in figure (#);-



i) When p/w is equal to (1) the response is a maximum. For curve A, μ is large and for curve C. μ is less. The response decreases for value of p/w greater than 1 or less than 1.

ii) When the frictional forces are absent, i.e., μ =0, R is infinite and the sharpness of resonance is maximum.

iii) The sharpness of resonance decreases with increase in the value of μ .

iv) The sharpness of resonance dies rapidly even for a very small change in the value of p/w from 1, in the case, where μ is minimum.

In the case of the resonance tube, the damping force is large and the graph will be similar to the curve (A).

3.6: Phase of Resonance:

Considering the phase lead of the forced vibrations with references to the deriving force in equation;

$$\tan\alpha = \frac{\mu\, p}{\mathrm{K} - mp^2}$$

$$\tan\alpha = \frac{\frac{\mu\, p}{m}}{(\frac{\mathrm{K}}{m} - p^2)}$$

At resonance ($\frac{\mathrm{K}}{m} = p^2$) and ($\tan\alpha = \mathrm{inf}\, inity$), i.e, $\alpha = \frac{\pi}{2}$

It means, for ($\frac{\mathrm{K}}{m} = 1$), ($\alpha = \frac{\pi}{2}$)

For value of ($\frac{\mathrm{K}}{m} > 1$), ($\alpha > \frac{\pi}{2}$)

For value of ($\frac{\mathrm{K}}{m} < 1$), ($\alpha < \frac{\pi}{2}$)

The graph for ($\frac{p}{w}$) along x-axis and (α) along of y-axis is shown in Figure (*).

The shape of the curve will also depend on the value of (μ), i.e., the external fractional forces.

(1) For values of (μ=0), the curve ABCDE represents the graph.

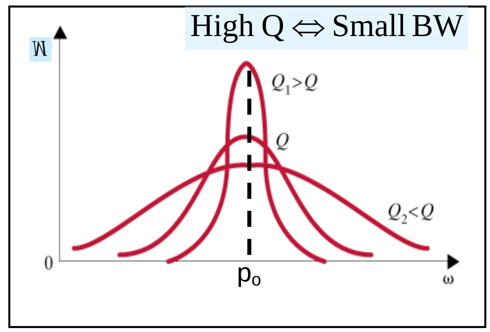
(2) For large values of (μ), the curve will be similar to curve 2.

(3) For small values of (μ), the curve will be similar to curve 1.

3.7: Quality Factor:

The sharpness of resonance is inversely proportional to μ and can be represented in terms of the quality factor. Suppose p_1 and p_2 are the values of the angular frequencies of the driving force on the two sides of the resonance angular frequency p_0 at which the power in the response is one half.

It is just the ratio of the resonance peak frequency to the peak width



The power at any instant is the product of the force and the velocity at that instant.

$$\therefore Average value of power = \frac{\mu F^2 p^2}{2[\mu^2 P^2 + (K - mP^2)^2]} ....(1)$$

The power will be maximum when $(k - mp^2) = o$

$$\therefore \max imum power = \frac{\mu F^2 p^2}{2\mu^2 p^2} = \frac{F^2}{2\mu} ....(2)$$

The average power will be half the maximum power when

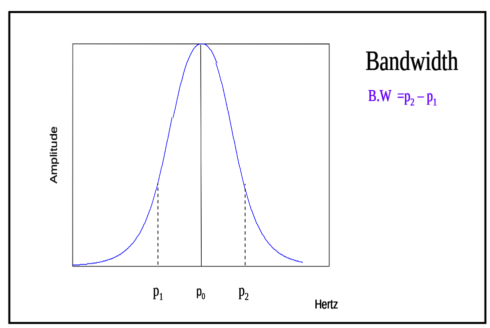
$$\mu^2 p^2 = (K - mp^2)^2 \qquad K - mP^2 = \pm \mu^2 p^2$$

or

$$k - mp^2 = \pm \mu \rho$$

$$mp^2 \pm \mu\rho - K = o \qquad p = \frac{\mu \pm \sqrt{\mu^2 + 4Km}}{2m}$$

one set of values of p_1 and p_2 are



$$p_1 = \frac{+\mu + \sqrt{\mu^2} + 4km}{2m} \qquad p_1 = \frac{-\mu + \sqrt{\mu^2} + 4km}{2m}$$

$$\therefore \qquad p_1 - p_2 = \frac{2\mu}{2\mu} = \frac{\mu}{m}$$

The quality factor $$Q = \frac{p_0}{p_1 - p_2} = \frac{\sqrt{k/m}}{\mu/m} = \frac{\sqrt{Km}}{\mu} ....(3)$$

3.8. Examples of forced and resonant vibration

1. An interesting example of forced vibrations and resonance is noticed in the case of troops marching over a suspension bridge . the marching troops exert a **periodic force** on the bridge and the bridge is set into forced vibrations . if all the soldiers march in step , each will exert a periodic force in phase and this results in the increased swing of the bridge . if however the natural frequency of the swing coincides with the forced frequency of the marching force , resonance occurs and the amplitude of the swing becomes very high .due to this reason it is dangerous and the bridge may give way . cones-quently the troops while crossing the bridge are usually ordered to march out of step.
2. A child exerts a periodic force to set the swing to move to and fro .the child also knows to give the push to the swing at periodic intervals.