Chapter Three

Free, Forced & Resonant Vibrations

Free, Forced & Resonant Vibrations

- When the motion is maintained by the restoring forces only, the vibration is described as *free vibration*. When a periodic force is applied to the system, the motion is described as *forced vibration*.
- When the frictional dissipation of energy is neglected, the motion is said to be *undamped*. Actually, all vibrations are *damped* to some degree.
- Examples of damping forces:
  - internal forces of a spring,
  - viscous force in a fluid,
  - electromagnetic damping in galvanometers,
  - shock absorber in a car.

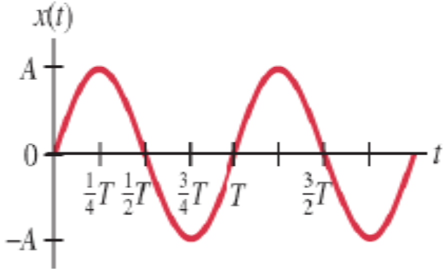
3.1: Free Vibrations:

- Vibrate in the absence of damping and external force. For example S.H.M.
- Characteristics:
  - the system oscillates with constant frequency and amplitude
  - the system oscillates with its natural frequency
  - the total energy of the oscillator remains constant

3.1: Free Vibrations:

- When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion.
- Time period (T) depends only on the length (l) and the acceleration due gravity (g).
- The pendulum will continue to oscillate with the same time period and amplitude for nay length of time.
- In such case, there is no loss of energy by fraction or otherwise.
- In all similar case, the vibrations will be undamped free vibration. The amplitude of swing remains constant.

$$T = 2\pi\sqrt{\frac{l}{g}}$$



3.2: Undamped Vibrations:

- For S.H.M., the kinetic energy for displacement y, is given by; $\frac{1}{2} m (\frac{dy}{dt})^2$
- and the potential energy i $\frac{1}{2} k y^2$
- The ***total energy*** at any instant, $= \frac{1}{2} m (\frac{dy}{dt})^2 + \frac{1}{2} k y^2$

$$\frac{1}{2} m (\frac{dy}{dt})^2 + \frac{1}{2} k y^2 = cons\tan t \quad ------ (1)$$

$$m (\frac{d^2 y}{dt^2}) + k y = 0 \quad ------ (2)$$

$$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0 \quad ------ (3)$$

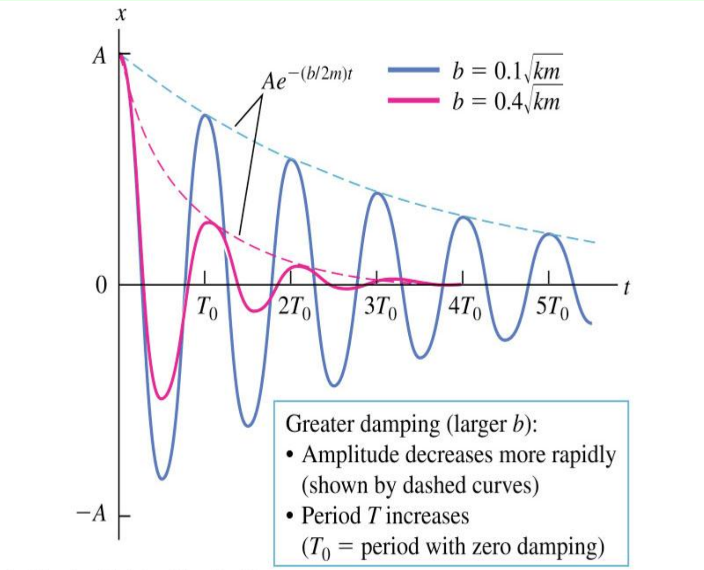
$$w^2 = \frac{k}{m}$$

$$\frac{d^2 y}{dt^2} + w^2 y = 0 \quad ------ (4)$$

$$y = a \sin (\sqrt{\frac{k}{m}} t - \alpha)$$

3.3: Damped Vibrations:

- The amplitude of swing decreases continuously with time and finally the oscillations die out.
- The oscillating system is opposed by dissipative forces.[When the pendulum vibrates in air medium, there are fictional forces and consequently is dissipated in each vibration.]
- The system does positive work on the surroundings.
- Examples:
  - a mass oscillates under water
  - oscillation of a metal plate in the magnetic field
- Total energy of the oscillator decreases with time
- The rate of loss of energy depends on the instantaneous velocity
- Frequency of damped vibration < Frequency of undamped vibration



3.3: Damped Vibrations:

- Let $\mu \frac{dy}{dt}$ be the dissipated force due to fraction.

$$m\frac{d^2y}{dt^2} + K\,y + \mu\frac{dy}{dt} = 0 \quad ------ (6)$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{m}\frac{dy}{dt} + \frac{K}{m}y = 0 \quad ------ (7)$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + w_o{}^2\,y = 0 \quad ------ (8)$$

$$y = a\,e^{-bt}\sin(wt - \alpha) \quad ------ (9)$$

$$y = Ae^{(-b+\sqrt{b^2 - w_o{}^2})t} + (-B)\,e^{(-b-\sqrt{b^2 - w_o{}^2})t} \quad ------ (10)$$

$$w = \sqrt{\frac{K}{m} - \frac{\mu^2}{4m^2}}$$

$$w_o{}^2 = \frac{K}{m}$$

$$2b = \frac{\mu}{m} \rightarrow \therefore b = \frac{\mu}{2m}$$

$$w = \sqrt{w_o{}^2 - b^2}$$

$$\omega' = \sqrt{\left(\frac{k}{m} - \frac{\mu^2}{4m^2}\right)} = \sqrt{\left(\omega_0{}^2 - \frac{\mu^2}{4m^2}\right)}$$

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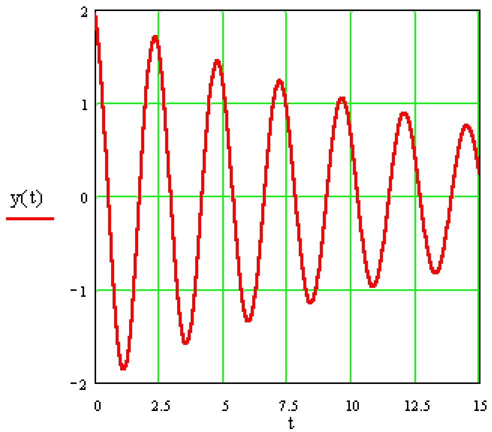
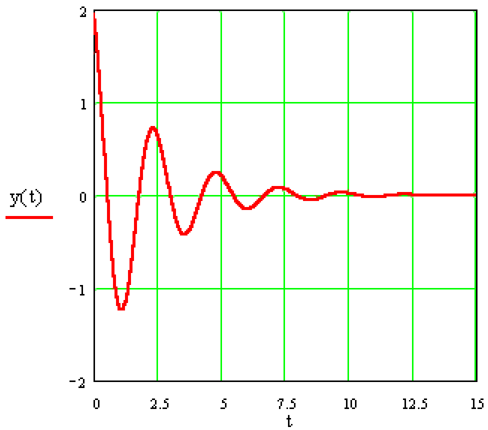
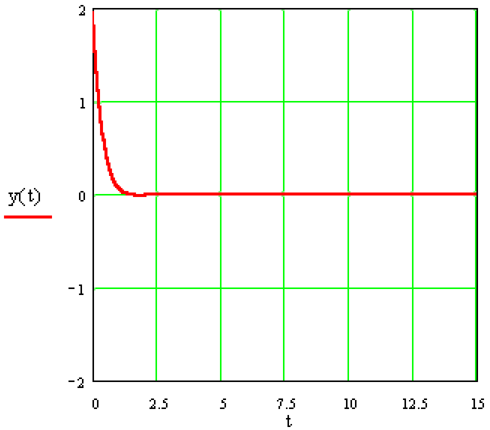


Light Damping
(small b/m)



Heavy Damping
(large b/m)



Critical Damping

$$\frac{k}{m} = \frac{b^2}{4m^2}$$

3.4: Forced Vibration:

- If some external period force is constantly applied on the body, the body continues to oscillate under the influence of such external forces; such vibrations of the body are called ***forced vibrations.***
- The frequency of the forced vibration is different from the natural frequency of vibration of the body.
- The ***amplitude*** of the body depends on the differences between the ***natural frequency*** and the ***frequency*** of the applied force. The amplitude will be large if difference in frequency is small and vice versa.

3.4: Forced Vibration:

- For forced vibration, equation (6) $m\frac{d^2y}{dt^2}+Ky+\mu\frac{dy}{dt}=0 \quad ------(6)$

$$m\frac{d^2y}{dt^2}+\mathrm{K}y+\mu\frac{dy}{dt}=F\sin pt \quad -----(11)$$

$$y=a\sin(pt-\alpha) \quad ------(12)$$

$$\frac{dy}{dt}=a\,p\cos(pt-\alpha) \quad -----(13)$$

$$\frac{d^2y}{dt^2}=-a\,p^2\sin(pt-\alpha)=-p^2y \quad ------(14)$$

$$-ma\,p^2[\sin pt\cos\alpha-\cos pt\sin\alpha]+\mathrm{K}a[\sin pt\cos\alpha-\cos pt\sin\alpha]$$
$$+\mu a\,p[\cos pt\cos\alpha+\sin pt\sin\alpha]-F\sin pt=0 \quad ------(15)$$

$$-ma\,p^2\cos\alpha+\mathrm{K}a\cos\alpha+\mu a\,p\sin\alpha-F=0 \quad ------(16)$$

sin pt =1 ; cos pt = 0

$$ma\,p^2\sin\alpha-\mathrm{K}a\sin\alpha+\mu a\,p\cos\alpha=0 \quad ------(17)$$

sin pt = 0 ; cos pt =1

$$\tan\alpha=\frac{\mu\,p}{(\mathrm{K}-m\,p^2)}=\frac{A}{B} \quad ------(18)$$

$$a=\frac{F}{\sqrt{\mu^2p^2+(\mathrm{K}-mp^2)^2}} \quad -------(21)$$

$$y=\frac{F}{\sqrt{\mu^2p^2+(\mathrm{K}-mp^2)^2}}\sin(pt-\alpha) \quad ------(22)$$

another solution is obtained when F=0. This corresponds to free vibrations.

$$y=a\,e^{-bt}\sin(wt-\alpha) \quad ------(23)$$

The general solution will include both the particular solutions for free and forced vibrations.

$$\therefore y=a\,e^{-bt}\sin(wt-\alpha)+\frac{F}{\sqrt{\mu^2p^2+(\mathrm{K}-mp^2)^2}}\sin(pt-\alpha)$$

3.5: Resonance and Sharpness of Resonance:

$$y = a e^{-bt} \sin(wt - \alpha) + \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha)$$

- If the effect of viscosity of the medium is small,
- the amplitude $\dfrac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}}$

Under the action of the driving force is a maximum when the denominator is a minimum. $K - mp^2 = 0 \rightarrow K = mp^2 \; or \; p = \sqrt{\dfrac{K}{m}}$

- Further, the amplitude will be infinite if is also *zero*. The oscillations will have maximum amplitude and this state of vibration of a system is called ***resonance***.
- If means that, when the ***forced frequency*** is ***equal*** to the ***natural frequency*** of vibration of the body, ***resonance*** takes place.

- If *friction* is present, the amplitude at resonance $= \dfrac{F}{\mu p} = \dfrac{F}{\mu \sqrt{K / m}}$ or $= \dfrac{F}{\mu} \sqrt{\dfrac{m}{K}}$

In the case of sound Resonance and Sharpness of Resonance

- In the case of sound, the study of sharpness of resonance is of great importance. Sharpness of resonance refers to the fall in amplitude with change in frequency on each side of the maximum amplitude.

$$y = \frac{F}{\sqrt{\mu^2 p^2 - (K + mp^2)^2}} \sin(pt - \alpha) \quad ----(1)$$

$$\frac{dy}{dt} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \cos(pt - \alpha) \quad ----(2)$$

$$\left(\frac{dy}{dt}\right)_{max} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad ----(3)$$

$$k.E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2_{max} = \frac{\frac{1}{2} m F^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \quad ----(4)$$

The *mean square* of the driving *force per unit mass*

$$= \frac{[\frac{0 + F^2}{2}]}{m} = \frac{F^2}{2m} \quad ------(5)$$

$$R = \frac{\frac{1}{2} m F^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \div \frac{F^2}{2m}$$

$$R = \frac{m^2 p^2}{\mu^2 p^2 + (K - mp^2)^2}$$

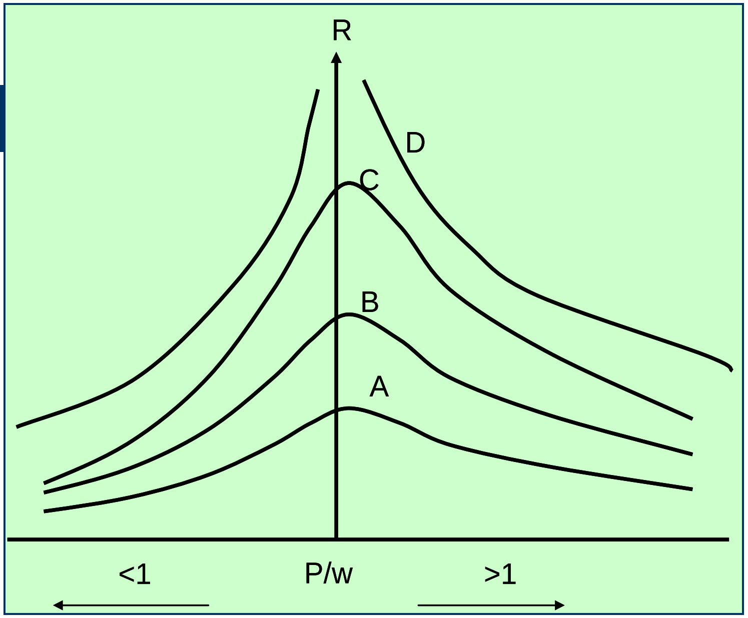
$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2} + (\frac{K}{m} - p^2)^2} \quad ----(6)$$

- The *natural frequency* of the system in the absence of damping is $\sqrt{\frac{K}{m}}$
- the nature frequency of the system deviates from the forced frequency.. $\frac{K}{m} - p^2$

$$\frac{K}{m} = p^2$$

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2}} = \frac{m^2}{\mu^2} = (\frac{m}{\mu})^2$$

$$R \alpha \frac{1}{\mu}$$



3.6: Phase of Resonance:

$$\tan\alpha = \frac{\mu\, p}{K - mp^2}$$

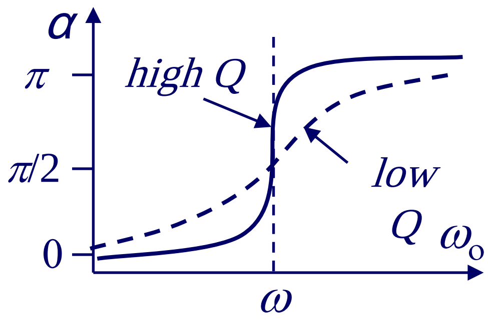
$$\tan\alpha = \frac{\frac{\mu\, p}{m}}{(\frac{K}{m} - p^2)}$$

$$\frac{K}{m} = p^2 \qquad \alpha = \frac{\pi}{2}$$

$$\frac{K}{m} = 1 \qquad \alpha = \frac{\pi}{2}$$

$$\frac{K}{m} > 1 \qquad \alpha > \frac{\pi}{2}$$

$$\frac{K}{m} < 1 \qquad \alpha < \frac{\pi}{2}$$



3.7: Quality Factor:

- The sharpness of resonance is inversely proportional to μ and can be represented in terms of the quality factor. Suppose p_1 and p_2 are the values of the angular frequencies of the driving force on the two sides of the resonance angular frequency p_0 at which the power in the response is one half.
- It is just the ratio of the resonance peak frequency to the peak width
- The power at any instant is the product of the force and the velocity at that instant.

$$\therefore Average value of power = \frac{\mu F^2 p^2}{2[\mu^2 P^2 + (K - mP^2)^2]} \text{.....................}(1)$$

$$(k - mp^2) = o$$

$$\therefore \max imum power = \frac{\mu F^2 p^2}{2\mu^2 p^2} = \frac{F^2}{2\mu} \text{.....................}(2)$$

$$\mu^2 p^2 = (K - mp^2)^2 \qquad K - mP^2 = \pm \mu^2 p^2$$

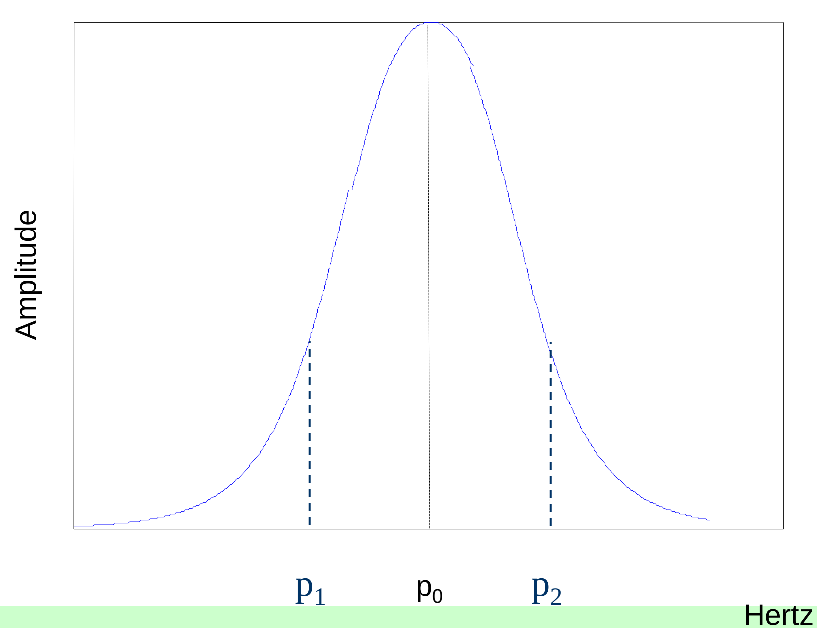
$$mp^2 \pm \mu\rho - K = o$$

$$p = \frac{\mu \pm \sqrt{\mu^2 + 4Km}}{2m}$$

$$p_1 = \frac{+\mu + \sqrt{\mu^2} + 4km}{2m}$$

$$p_2 = \frac{-\mu + \sqrt{\mu^2} + 4km}{2m}$$

$$Q = \frac{p_0}{p_1 - p_2} = \frac{\sqrt{k/m}}{\mu/m} = \frac{\sqrt{Km}}{\mu} \text{.....................}(3)$$

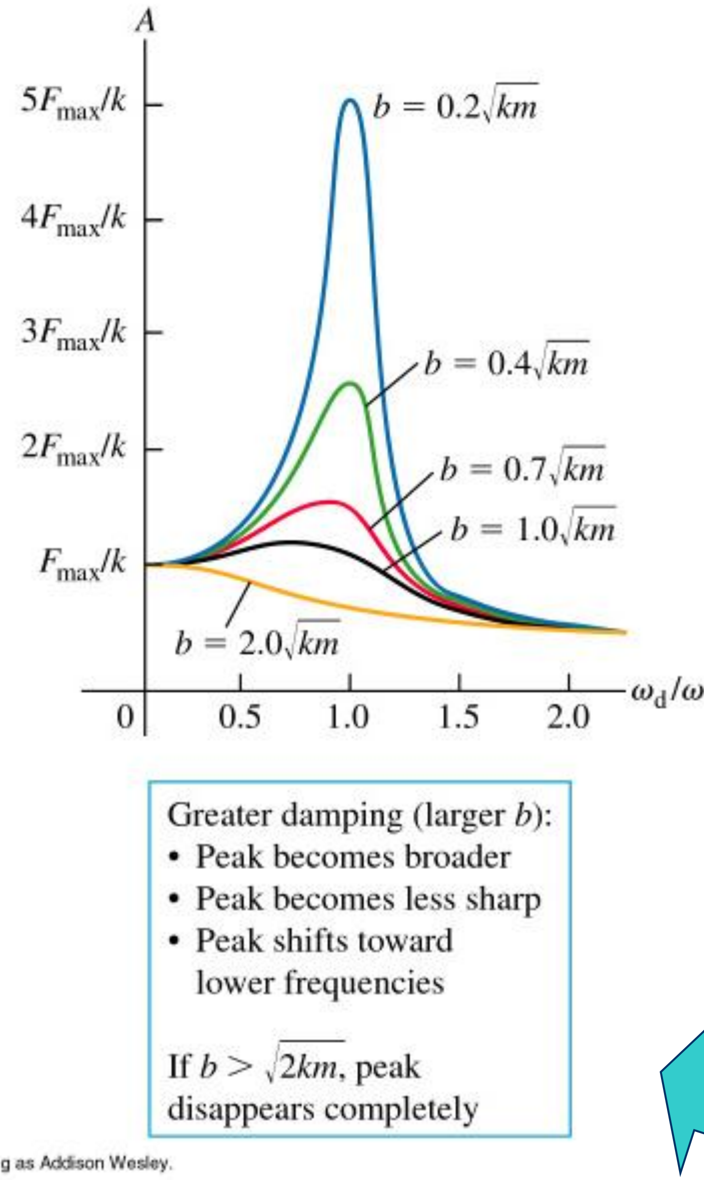


Bandwidth

$B.W = p_2 - p_1$

$$P_2 - p_1 = \frac{2\mu}{2\mu} = \frac{\mu}{m}$$

3.7: Quality Factor:



When driving frequency = natural frequency of oscillator, amplitude is maximum.

We say the system is in RESONANCE

“Sharpness” of resonance peak described by quality factor (Q)

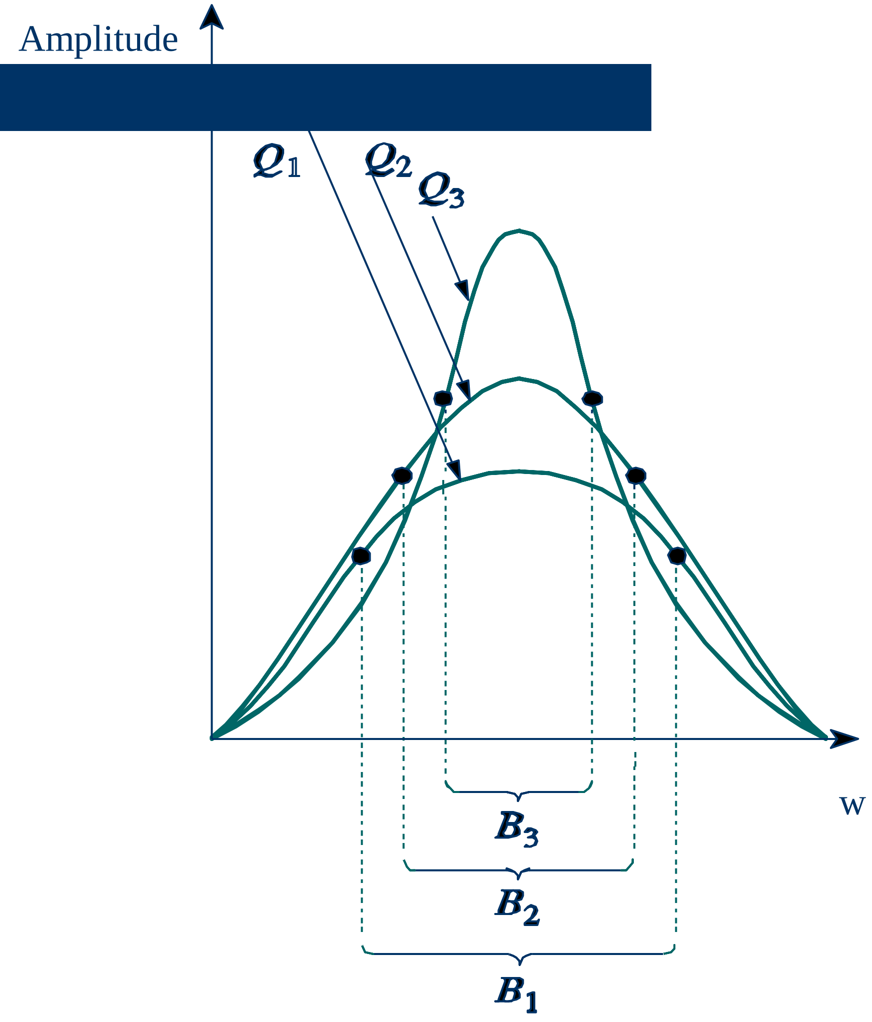
High Q = sharp resonance

Damping reduces Q

Quality Factor

The Quality Factor (Q) is the ratio of the resonant frequency to its bandwidth.

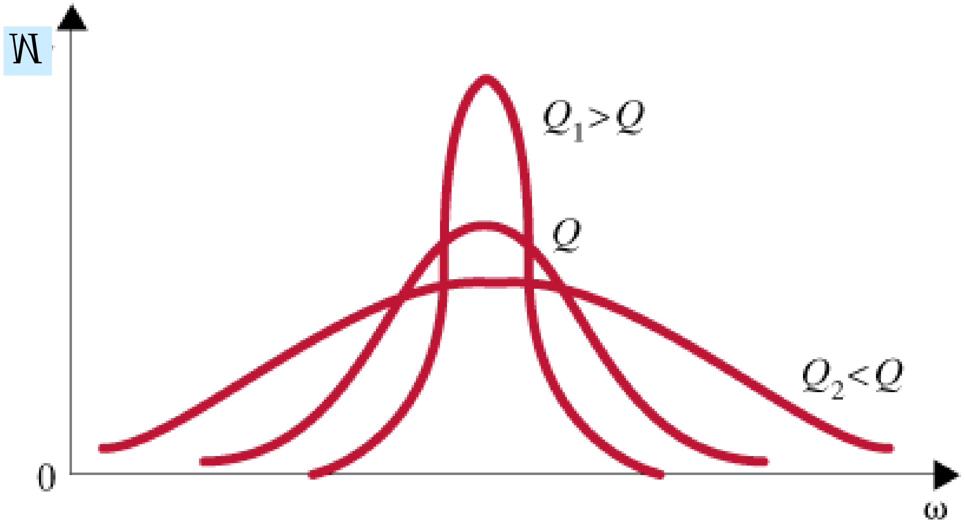
$$Q = \frac{\omega_o}{B}$$

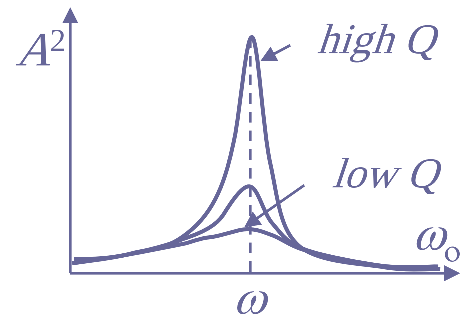


Resonance (1)

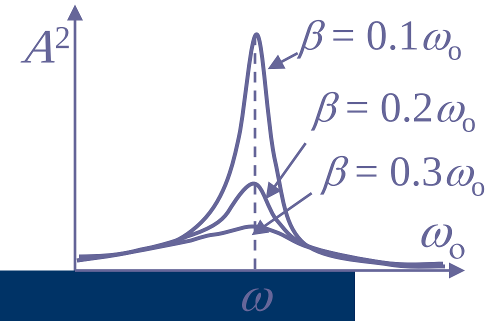
- Resonance occurs when an oscillator is acted upon by a second driving oscillator whose frequency ***equals*** the natural frequency of the system
- The amplitude of reaches a ***maximum***
- The energy of the system becomes a ***maximum***
- The phase of the displacement of the driver ***leads*** that of the oscillator by 90°

High Q ⇔ Small BW





We have met a number of frequencies in this discussion, so it might be helpful to list them in one place to help keep them straight.



$$\omega_o = \sqrt{k/m} = \text{natural frequency of undamped oscillator}$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \text{frequency of damped oscillator}$$

$$\omega = \text{frequency of driving force}$$

$$\omega_2 = \sqrt{\omega_0^2 - 2\beta^2} = \text{value of } \omega \text{ at which response is max}$$

To find out the maximum amplitude of a particular driven oscillator, just let *w* ~ *w*o in

i.e.

$$A^2 = \frac{f_o^2}{\left(\omega_o^2 - \omega^2\right)^2 + 4\beta^2\,\omega^2}.$$

From this you can see that the amplitude goes as *b*-1. It also turns out that the width of the resonance increases with *b* as FWHM ~ 2*b*. (Prob. 5.41)