1-9: Orthogonal Coordinate Systems: In electromagnetic, the physical quantities we deal with are, in general, functions of space and time. A three dimensional coordinate system allows us to uniquely specify the location of a point in space or the direction of a vector quantity. The most standard orthogonal coordinate systems that are commonly used are:

(1). Cartesian (or rectangular) Coordinate system (x, y, z)

(2). Cylindrical Coordinate system (ρ, ϕ, z)

(3). Spherical Coordinate system (r, θ, ϕ)

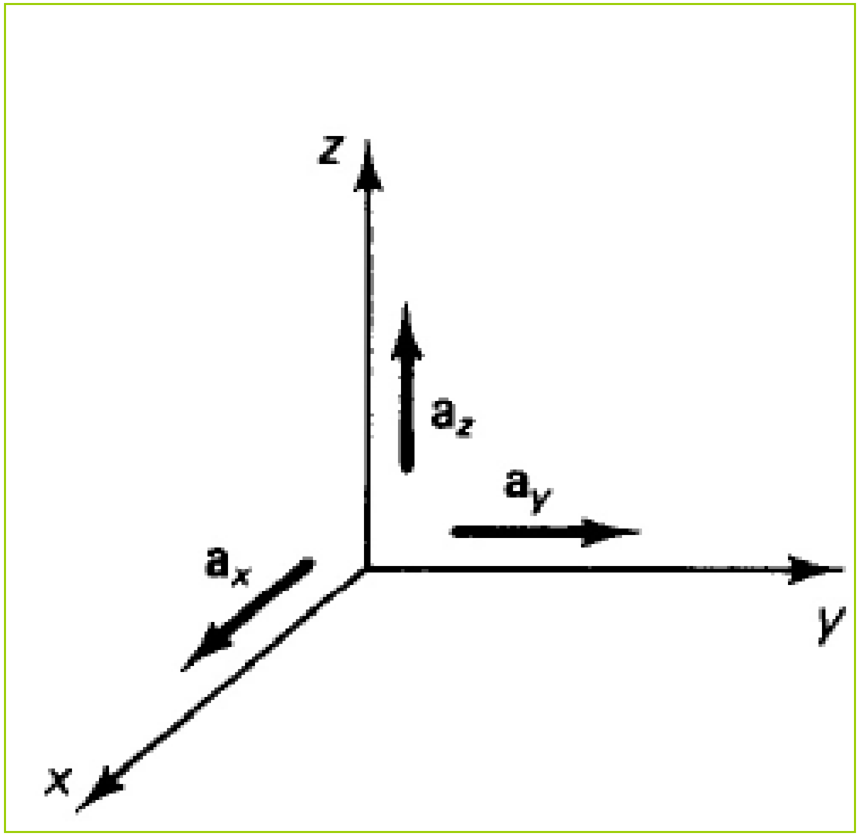
1-9-1: Cartesian coordinate system:

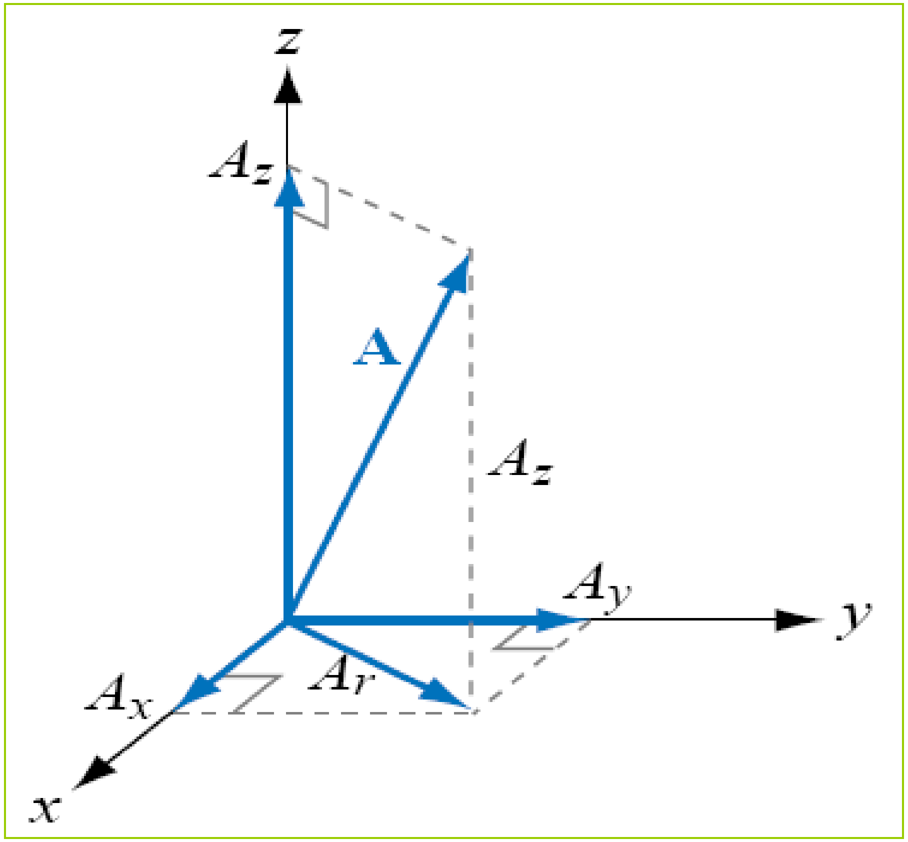
In Cartesian coordinate system, the vector, element length, surface and volumes are represented as given below.

$$-\infty \le x \le \infty \quad , \quad -\infty \le y \le \infty \quad , \quad -\infty \le z \le \infty$$

The vector is denoted by: $\vec{\mathbf{A}} = \mathbf{A}_x\,\hat{a}_x + \mathbf{A}_y\,\hat{a}_y + \mathbf{A}_z\,\hat{a}_z$

The length of a vector: $|\mathbf{A}| = \sqrt{\mathbf{A}_x^2 + \mathbf{A}_y^2 + \mathbf{A}_z^2}$





Element of length:

$$\vec{\mathbf{d}l} = \mathbf{d}l_x \ \hat{a}_x + \mathbf{d}l_y \ \hat{a}_y + \mathbf{d}l_z \ \hat{a}_z$$

$$\mathbf{d}l_x = dx \quad , \quad \mathbf{d}l_y = dy \quad , \quad \mathbf{d}l_z = dz$$

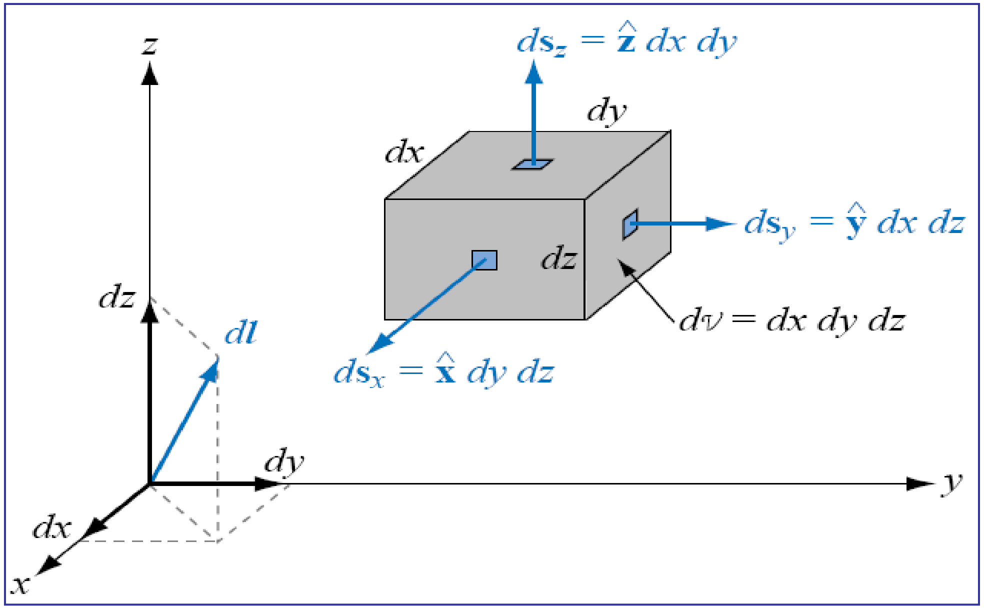
Element of Surface:

$$\vec{\mathbf{d}s} = ds_x \ \hat{a}_x + ds_y \ \hat{a}_y + ds_z \ \hat{a}_z$$

$$ds_x = dy\,dz \quad , \quad ds_y = dx\,dz \quad , \quad ds_z = dx\,dy$$

Element of Volume:

$$dv = dx\,dy\,dz$$



$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

$$\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z \qquad \hat{a}_y \times \hat{a}_z = \hat{a}_x \qquad \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_y \times \hat{a}_x = -\hat{a}_z \qquad \hat{a}_z \times \hat{a}_y = -\hat{a}_x \qquad \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

1-9-2: Cylindrical coordinate system:

In cylindrical coordinate system, the vectors, element length, surface and volumes are represented as given below.

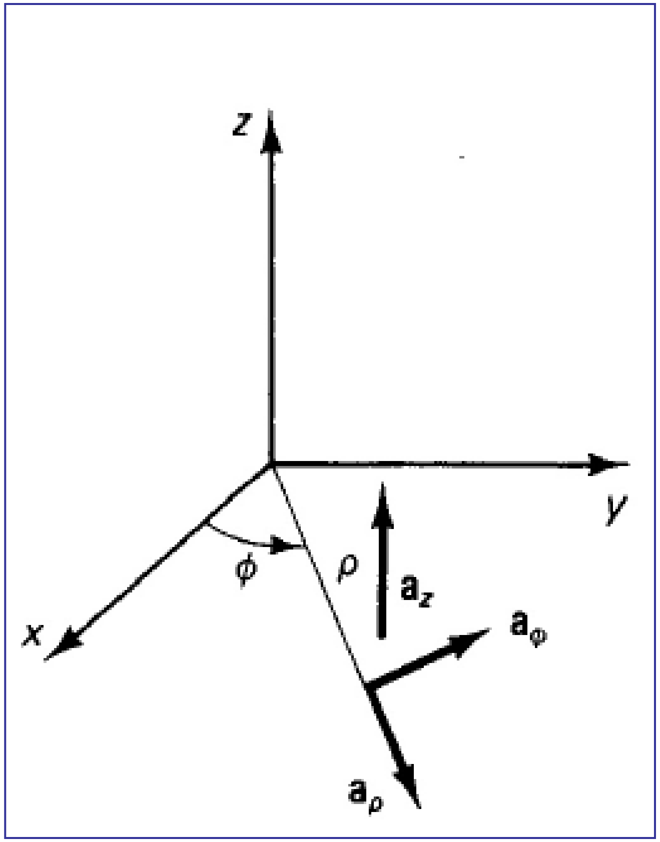
$$0 \le \rho \le \infty \;,\; 0 \le \phi \le 2\pi \;,\; -\infty \le z \le \infty$$

Vector is represented by:

$$\vec{\mathbf{A}} = \mathbf{A}_\rho \hat{a}_\rho + \mathbf{A}_\phi \hat{a}_\phi + \mathbf{A}_z \hat{a}_z$$

Length of vector is given by:

$$|\mathbf{A}| = \sqrt{\mathbf{A}_\rho^2 + \mathbf{A}_\phi^2 + \mathbf{A}_z^2}$$



The coordinates are related to Cartesian coordinates by the following relations:

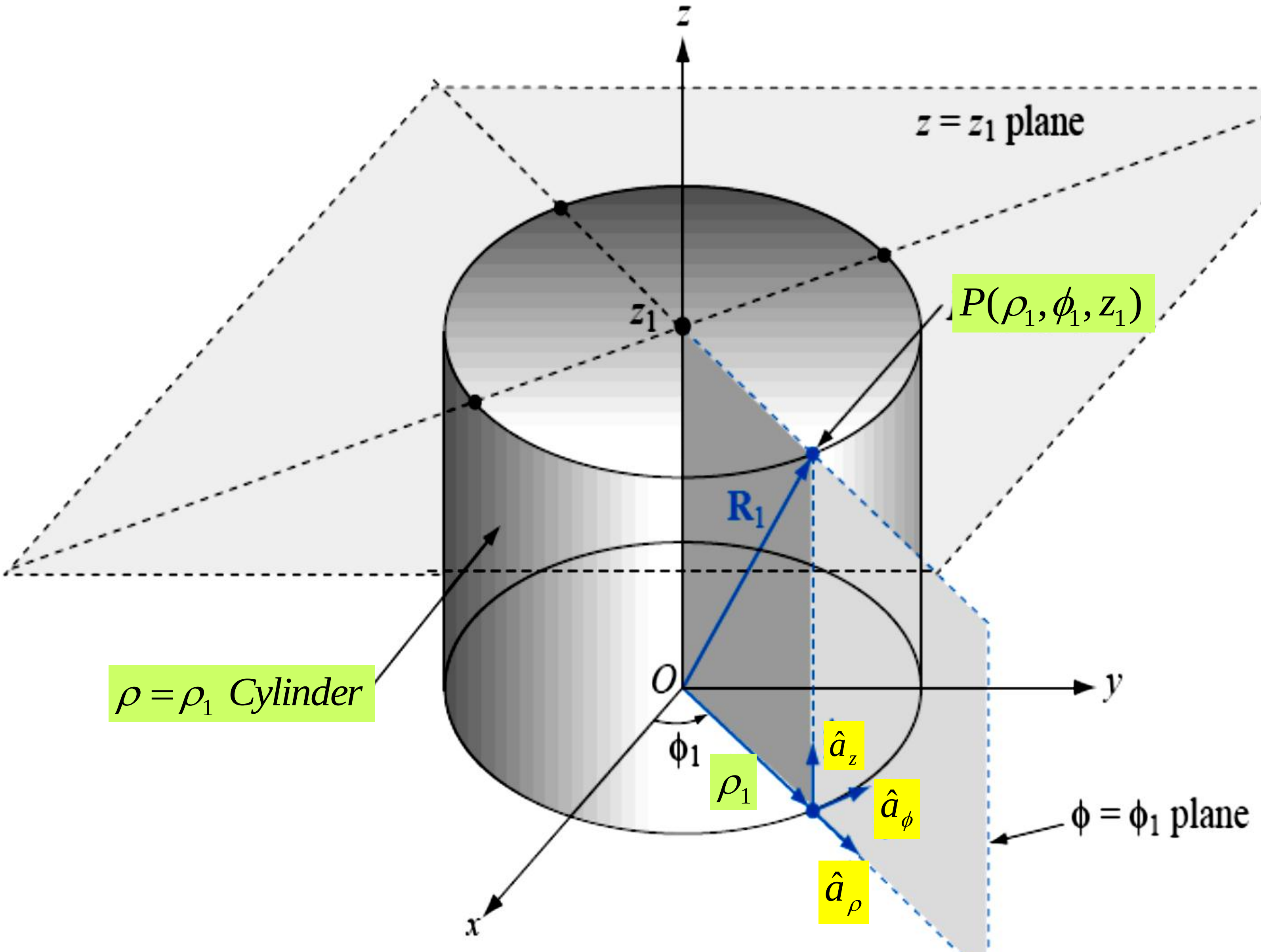
$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = \rho \cos\phi$$

$$y = \rho \sin\phi$$

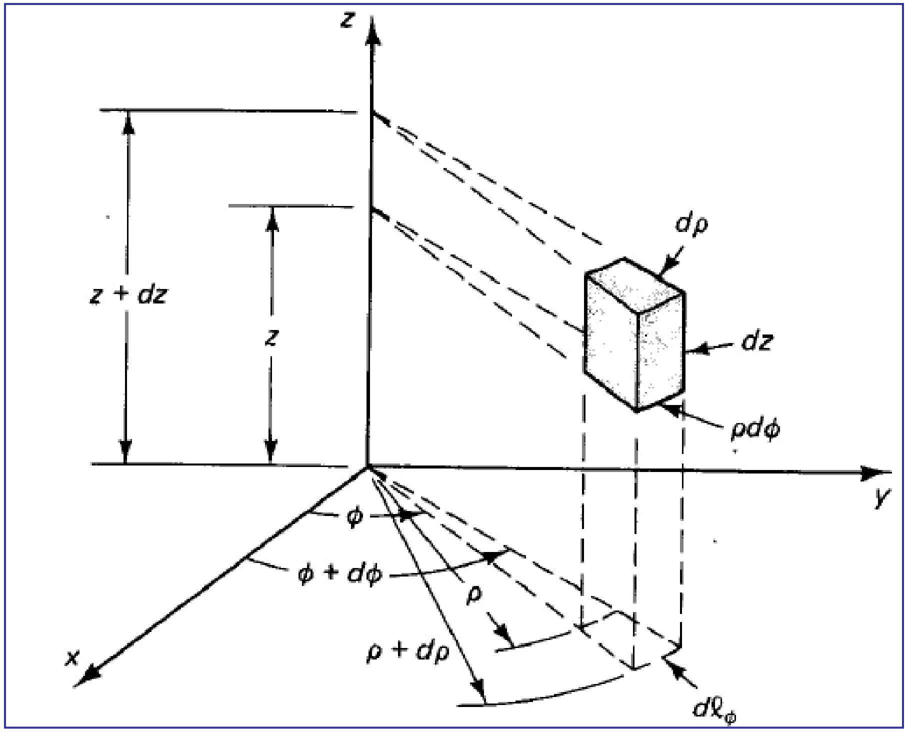
$$z = z$$


z
z = z1 plane
P(ρ1, ϕ1, z1)
z1
R1
ρ = ρ1 Cylinder
O
y
ϕ1
ρ1
âz
âϕ
âρ
ϕ = ϕ1 plane
x

Element of Length:

$$\vec{\mathbf{d}l} = \mathbf{d}l_\rho\ \hat{a}_\rho + \mathbf{d}l_\phi\ \hat{a}_\phi + \mathbf{d}l_z\ \hat{a}_z$$

$$\mathbf{d}l_\rho = d\rho \quad , \quad \mathbf{d}l_\phi = \rho\, d\phi \quad , \quad \mathbf{d}l_z = dz$$



Element of Surface:

$$\vec{\mathbf{d}s} = ds_\rho\ \hat{a}_\rho + ds_\phi\ \hat{a}_\phi + ds_z\ \hat{a}_z$$

$$ds_\rho = \rho\, d\phi\, dz \qquad \rho = \text{constant}$$

$$ds_\phi = d\rho\, dz \qquad \phi = \text{constant}$$

$$ds_z = \rho\, d\rho\, d\phi \qquad z = \text{constant}$$

$$\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot \hat{a}_\phi = \hat{a}_z \cdot \hat{a}_z = 1$$

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0$$

$$\hat{a}_\rho \times \hat{a}_\rho = \hat{a}_\phi \times \hat{a}_\phi = \hat{a}_z \times \hat{a}_z = 0$$

Element of Volume:

$$dv = \rho\ d\rho\, d\phi\, dz$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \qquad \hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho \qquad \hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$$

$$\hat{a}_\phi \times \hat{a}_\rho = -\hat{a}_z \qquad \hat{a}_z \times \hat{a}_\phi = -\hat{a}_\rho \qquad \hat{a}_\rho \times \hat{a}_z = -\hat{a}_\phi$$

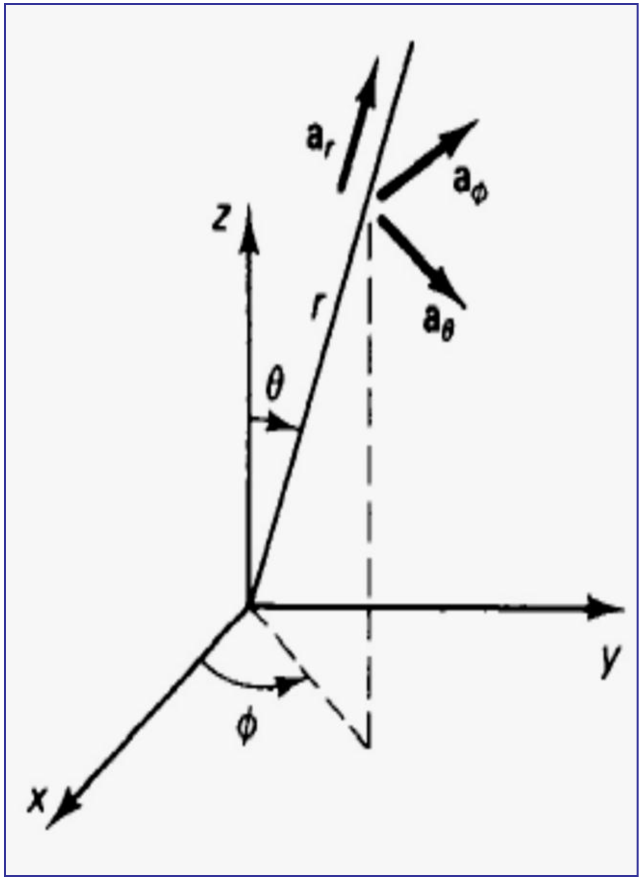
1-9-3: Spherical coordinate system:

In spherical coordinate system, the vectors, element length and surface vectors with volume element are represented as given below.

$$0 \le r \le \infty \;,\quad 0 \le \phi \le 2\pi \;,\quad 0 \le \theta \le \pi$$

Vector is represented by : $\vec{\mathbf{A}} = \mathbf{A}_r \, \hat{a}_r + \mathbf{A}_\theta \, \hat{a}_\theta + \mathbf{A}_\phi \, \hat{a}_\phi$

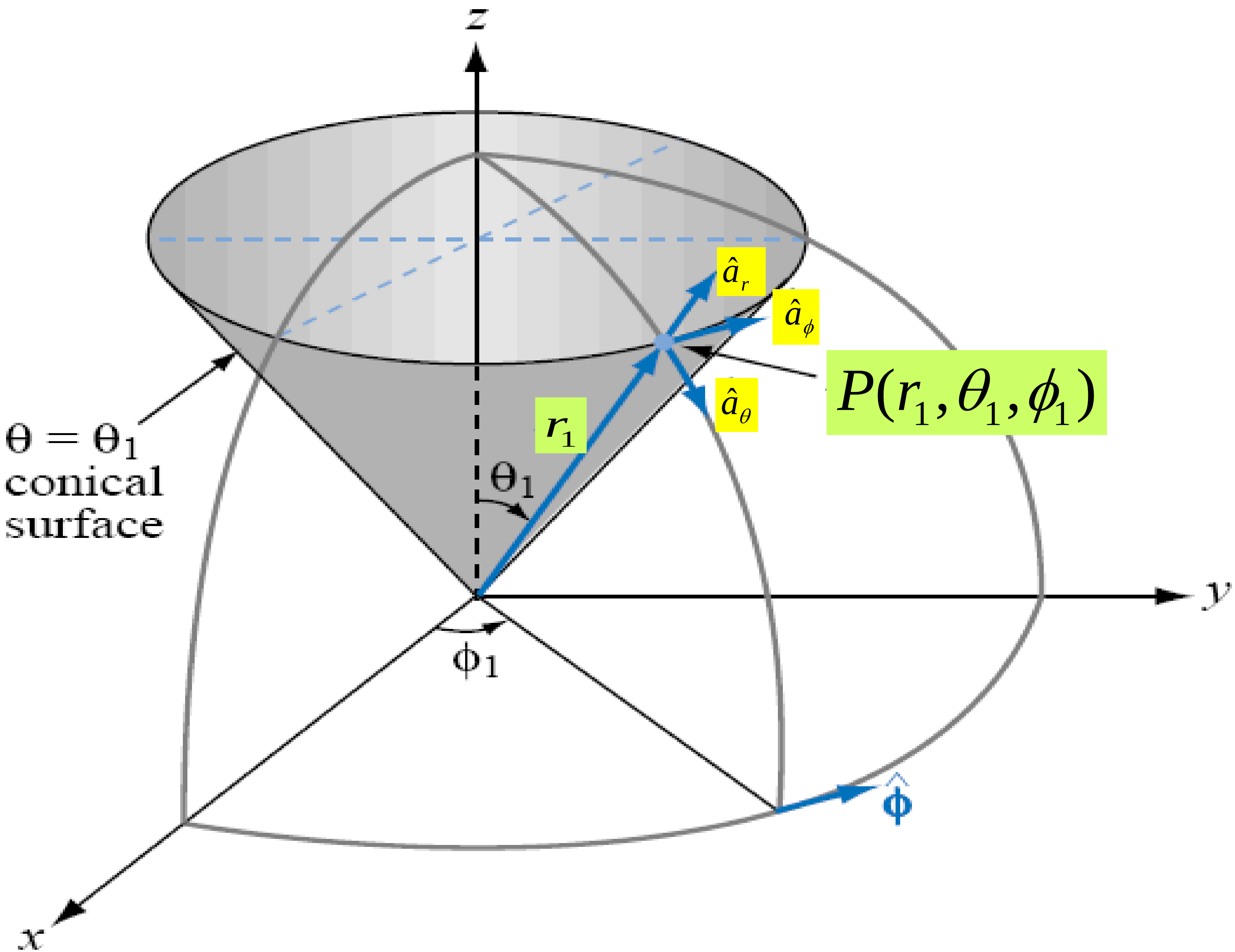
Length of vector is given by: $|\mathbf{A}| = \sqrt{\mathbf{A}_r^2 + \mathbf{A}_\theta^2 + \mathbf{A}_\phi^2}$

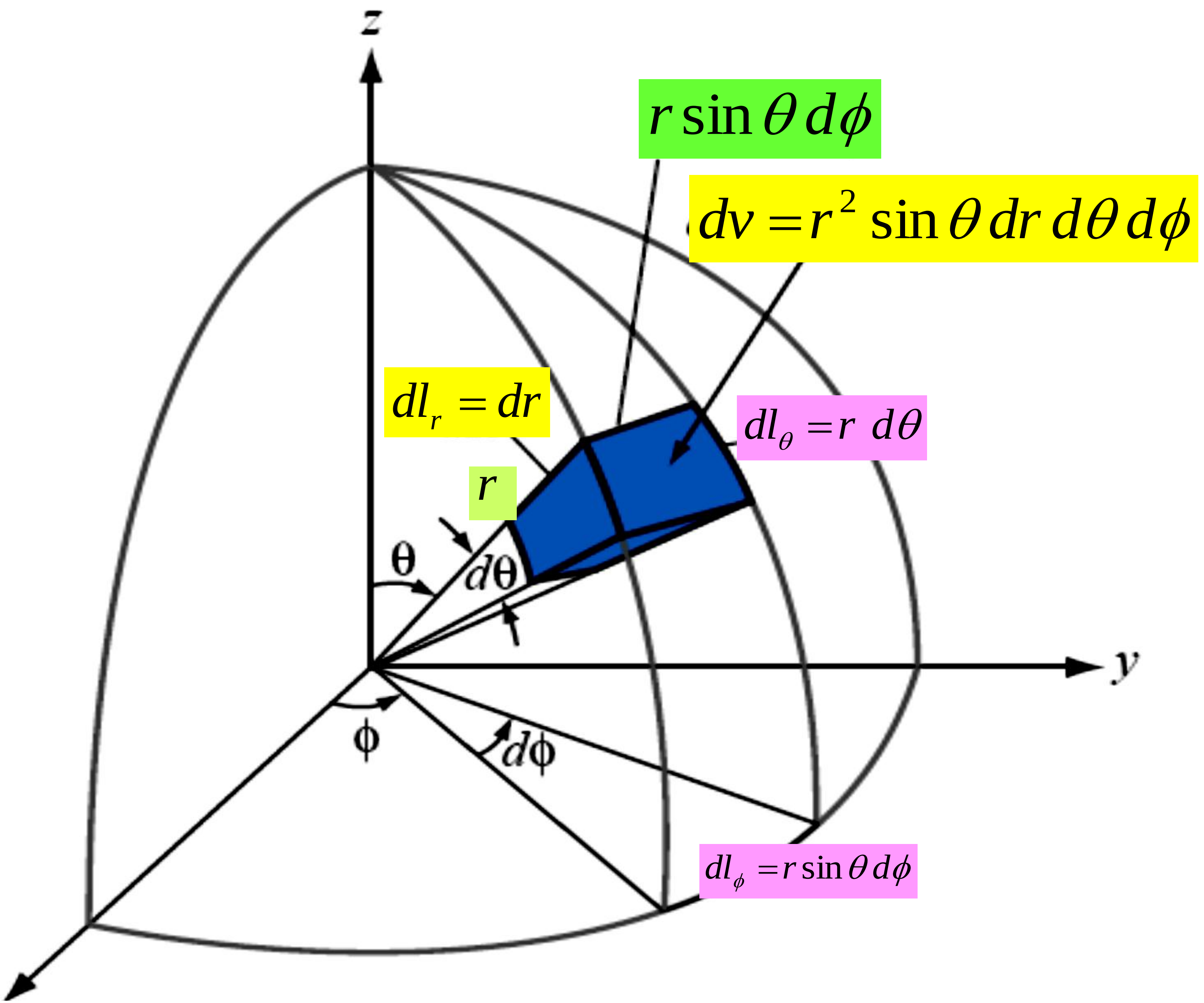


The spherical coordinates are related to Cartesian coordinates by the following relations:

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

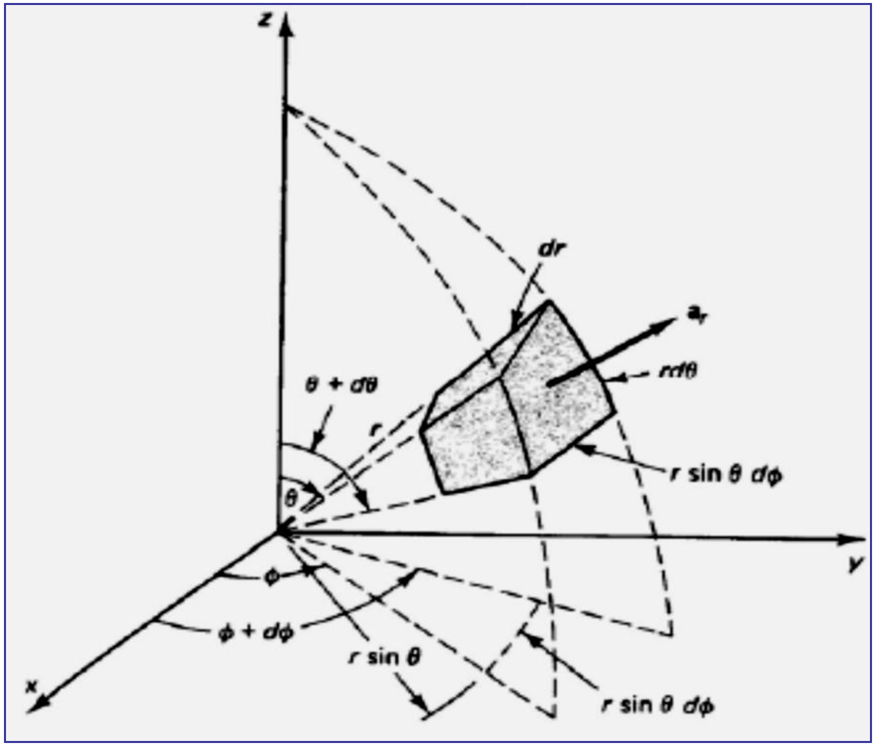

z
$\hat{a}_r$
$\hat{a}_\phi$
$P(r_1, \theta_1, \phi_1)$
$\hat{a}_\theta$
r_1
$\theta = \theta_1$
conical
surface
θ_1
y
ϕ_1
$\hat{\phi}$
x


z
$r\sin\theta\, d\phi$
$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$
$dl_r = dr$
$dl_\theta = r\ d\theta$
r
θ
dθ
y
φ
dφ
$dl_\phi = r\sin\theta\, d\phi$

Element of Length:

$$\overrightarrow{\mathbf{d}l} = \mathbf{d}l_r \ \hat{a}_r + \mathbf{d}l_\theta \ \hat{a}_\theta + \mathbf{d}l_\phi \ \hat{a}_\phi$$

$$\mathbf{d}l_r = dr \quad , \ \mathbf{d}l_\theta = r\, d\theta \quad , \quad \mathbf{d}l_\phi = r \sin\theta \ d\phi$$



Element of Surface:

$$\overrightarrow{\mathbf{d}s} = ds_r \ \hat{a}_r + ds_\theta \ \hat{a}_\theta + ds_\phi \ \hat{a}_\phi$$

$$ds_r = r^2 \sin\theta \ d\theta\, d\phi \quad , \quad r = cons\tan t$$

$$ds_\theta = r \sin\theta\, dr\, d\phi \quad , \quad \theta = cons\tan t$$

$$ds_\phi = r\, dr\, d\theta \quad , \quad \phi = cons\tan t$$

Element of Volume:

$$dv = r^2 \sin\theta \ dr\, d\theta\, d\phi$$

$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot \hat{a}_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1$$

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$$

$$\hat{a}_r \times \hat{a}_r = \hat{a}_\theta \times \hat{a}_\theta = \hat{a}_\phi \times \hat{a}_\phi = 0$$

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi \qquad \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r \qquad \hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$

$$\hat{a}_\theta \times \hat{a}_r = -\hat{a}_\phi \qquad \hat{a}_\phi \times \hat{a}_\theta = -\hat{a}_r \qquad \hat{a}_r \times \hat{a}_\phi = -\hat{a}_\theta$$

Example(1): Find the distance between the following pairs of points :

(a)- P_1 (1,1,2) and P_2 (0,2,2) **(b). P_1 (2,** $\frac{\pi}{3}$,1) *and* $P_2(4,\frac{\pi}{2},0)$ **(c).** $P_1(3,\pi,\frac{\pi}{2})$ *and* $P_2(4,\frac{\pi}{2},\pi)$

Solution:

(a). $$\overline{P_1P_2}=\sqrt{(1-0)^2+(2-1)^2+(2-2)^2}=\sqrt{2}\ unit$$

(b).

$$x_{p1}=\rho_1\cos\phi_1=2\cos 60=1 \qquad y_{p1}=\rho_1\sin\phi_1=2\sin 60=1.73 \qquad z_{p1}=1$$

$$x_{p2}=\rho_2\cos\phi_2=4\cos 90=0 \qquad y_{p2}=\rho_2\sin\phi_2=4\sin 90=4 \qquad z_{p2}=0$$

$$\overline{p_1p_2}=\sqrt{(1-0)^2+(1.73-4)^2+(1-0)^2}=2.06\ unit$$

(c).

$$x_{p1}=r_1\sin\theta_1\cos\phi_1=3\sin\pi\cos 90=0 \qquad y_{p1}=r_1\sin\theta_1\sin\phi_1=2\sin\pi\sin 60=0 \qquad z_{p1}=r_1\cos\theta_1=3\cos\pi=-3$$

$$x_{p2}=r_2\sin\theta_2\cos\phi_2=4\sin 90\cos\pi=-4 \qquad y_{p2}=r_2\sin\theta_2\sin\phi_2=4\sin 90\sin\pi=0 \qquad z_{p2}=r_2\cos\theta_2=4\cos 90=0$$

$$\overline{p_1p_2}=\sqrt{(0-(-4))^2+(0-0)^2+(-3-0)^2}=\sqrt{16+9}=5\ unit$$

Example(2):Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates : (a)- P_1 (1,2,0) , (b)- P_2 (0,0,3) and (c)- P_3 (1,1,2)

Solution:

In general the Cartesian coordinates are related to cylindrical and spherical coordinates by the following equations:

$$\rho=\sqrt{x^2+y^2} \qquad \phi=\tan^{-1}\frac{y}{x} \qquad z=z$$

$$r=\sqrt{x^2+y^2+z^2} \qquad \phi=\tan^{-1}\frac{y}{x} \qquad \theta=\cos^{-1}\frac{z}{r}$$

Therefore these points in cylindrical coordinates are given as follows:

(a).

$$x_{P1}=1 \qquad y_{P1}=2 \qquad and \qquad z_{P1}=0$$

$$\therefore\ \rho_{P1}=\sqrt{x_{P1}{}^2+y_{P1}{}^2}=\sqrt{5} \qquad \phi_{P1}=\tan^{-1}\frac{y_{P1}}{x_{P1}}=\tan^{-1}\frac{2}{1}=63^\circ \qquad and \quad z_{P1}=0$$

$$P_1(\sqrt{5},63^\circ,0)$$

(b).

$$x_{P2}=0 \qquad y_{P2}=0 \qquad and \qquad z_{P2}=3$$

$$\therefore\ \rho_{P2}=\sqrt{x_{P2}{}^2+y_{P2}{}^2}=0 \qquad \phi_{P2}=\tan^{-1}\frac{y_{P2}}{x_{P2}}=\tan^{-1}\frac{0}{0}=90^\circ \qquad and \quad z_{P2}=3$$

$$P_2(0,90^\circ,3)$$

(c).

$$x_{P3}=1 \qquad y_{P3}=1 \qquad and \qquad z_{P3}=2$$

$$\therefore\ \rho_{P3}=\sqrt{x_{P3}{}^2+y_{P3}{}^2}=\sqrt{2} \qquad \phi_{P3}=\tan^{-1}\frac{y_{P3}}{x_{P3}}=\tan^{-1}\frac{1}{1}=45^\circ \qquad and \quad z_{P3}=2$$

$$P_3(\sqrt{2},45^\circ,2)$$

Also these points in spherical coordinates are given as follows:

(a).

$$x_{P1} = 1 \qquad y_{P1} = 2 \qquad and \qquad z_{P1} = 0$$

$$\therefore\ r_{P1} = \sqrt{x_{P1}{}^2 + y_{P1}{}^2 + z_{P1}{}^2} = \sqrt{5} \qquad \phi_{P1} = \tan^{-1}\frac{y_{P1}}{x_{P1}} = \tan^{-1}\frac{2}{1} = 63^\circ \qquad and \quad \theta_{P1} = \cos^{-1}\frac{0}{\sqrt{5}} = 90^\circ$$

$$P_1(\sqrt{5}, 63^\circ, 90^\circ)$$

(b).

$$x_{P2} = 0 \qquad y_{P2} = 0 \qquad and \qquad z_{P2} = 3$$

$$\therefore\ \rho_{P2} = \sqrt{x_{P2}{}^2 + y_{P2}{}^2 + z_{P2}{}^2} = 3 \qquad \phi_{P2} = \tan^{-1}\frac{y_{P2}}{x_{P2}} = \tan^{-1}\frac{0}{0} = 90^\circ \qquad and \quad \theta_{P2} = \cos^{-1}\frac{3}{3} = 0^\circ$$

$$P_2(3, 90^\circ, 0^\circ)$$

(c).

$$x_{P3} = 1 \qquad y_{P3} = 1 \qquad and \qquad z_{P3} = 2$$

$$\therefore\ \rho_{P3} = \sqrt{x_{P3}{}^2 + y_{P3}{}^2 + z_{P3}{}^2} = \sqrt{6} \qquad \phi_{P3} = \tan^{-1}\frac{y_{P3}}{x_{P3}} = \tan^{-1}\frac{1}{1} = 45^\circ \qquad and \quad \theta_{P3} = \cos^{-1}\frac{2}{\sqrt{6}} = 35.26$$

$$P_3(\sqrt{6}, 45^\circ, 35.26^\circ)$$

Home Work

Q1/ Find the distance between the following pair of points:

(a). $(1, \frac{\pi}{6}, 0)$ *and* $(1, \pi, 2)$

(b). P1$(2, \frac{\pi}{3}, 1)$ *and* $P_2(4, \frac{\pi}{2}, 0)$

Q2/ A field is expressed in spherical coordinate by $\vec{A} = \frac{100}{r^2}\hat{a}_r$ find the magnitude of the field at point (-3,4,10) and determine the angle it makes with the vector $\vec{D} = 2\hat{a}_x - 2\hat{a}_y + \hat{a}_z$

Q3 / Convert the coordinates of the following points from spherical to Cartesian and cylindrical coordinates: $a - P_1(5, 0^\circ, 0^\circ)$, $b - P_2(5, 0^\circ, \pi)$ *and* $c - P_3(3, \frac{\pi}{2}, \pi)$

Q4/ Given vectors : $\vec{A} = (\cos\phi + 3z)\hat{a}_\rho - (2\rho + 4\sin\phi)\hat{a}_\phi + (\rho - 2z)\hat{a}_z$ *and* $\vec{B} = -\sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_z$

Find:

(a).The angle between $\vec{A}$ *and* $\vec{B}$ *at* $(2, 90^\circ, 0)$

(b). A unit vector perpendicular to both . $\vec{A}$ *and* $\vec{B}$ *at point* $(2, 90^\circ, 1)$

Q5/Given vectors $\bar{A}=(\cos\varphi+3z)\hat{a}_\rho-(2\rho+4\sin\varphi)\hat{a}_\varphi+(\rho-2z)\hat{a}_z \quad and \quad \vec{B}=-\sin\varphi\hat{a}_\rho+\cos\varphi\hat{a}_z$

, then find the following: (a)- θ_{AB}

(b)- A unit vector perpendicular to both $\vec{A}$ *and* $\vec{B}$ *at point* $(2,\frac{\pi}{2},1)$

Q6/ Find the vector directed from $(10,\frac{3\pi}{4},\frac{\pi}{6})$ *to* $(5,\frac{\pi}{4},\pi)$**, where the point are given in spherical coordinates?**

Q7/ Verify that the distance (d) between the two points in cylindrical coordinate (ρ_1,φ_1,z_1) **and** (ρ_2,φ_2,z_2) **is given by : .**

$$d^2=\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\varphi_2-\varphi_1)+(z_2-z_1)^2$$