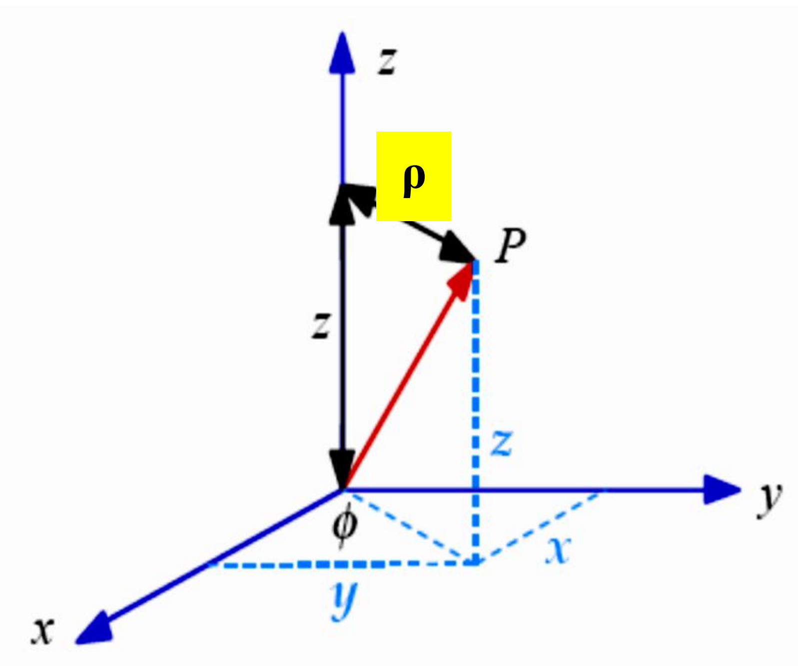
Chapter Two
Vector Calculus and Transformation Between Coordinate Systems

2-1:Transformation Between Coordinate Systems:

The position of a given point in space is invariant with respect to the choice of coordinate system. That is, its location is the same irrespective of which specific coordinate system is used to represent it. The same is true for vectors. The relation between the variables (x, y, z) -Cartesian (ρ, ϕ, z) Cylindrical and (r, θ, ϕ) in Spherical coordinates will be established to transform any one of the three system into vectors expressed in any of the other two.

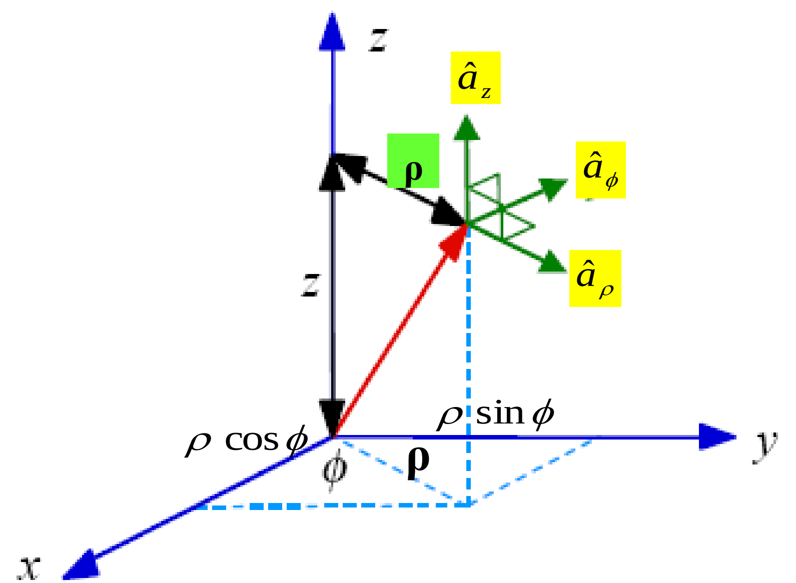
2-1-1: Cartesian to Cylindrical Transformation:

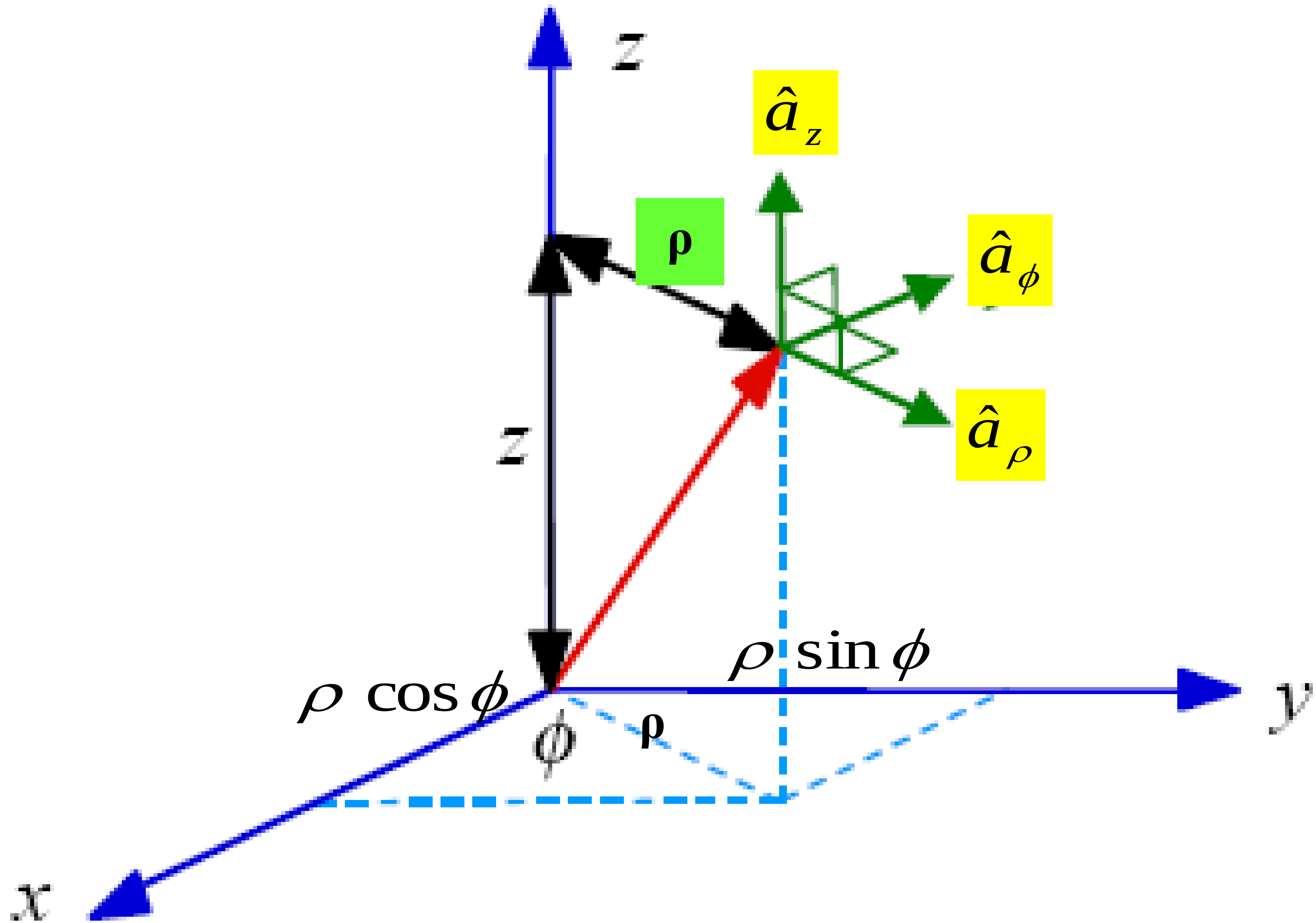
Point **P** in the figure below , has Cartesian coordinate (x, y, z) and cylindrical coordinate (ρ, ϕ, z) . Both systems share the coordinate (**z** **),** and the relation between the other two pairs of coordinates can be obtained as follows:



$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$





z
$\hat{a}_z$
ρ
$\hat{a}_\phi$
$\hat{a}_\rho$
z
$\rho \sin\phi$
$\rho \cos\phi$
y
ϕ
ρ
x

The relation between the unit vectors of Cartesian and Cylindrical coordinates are given by:

$$\hat{a}_x \cdot \hat{a}_\rho = \cos\phi \qquad \hat{a}_y \cdot \hat{a}_\rho = \sin\phi \qquad \hat{a}_z \cdot \hat{a}_\rho = 0$$

$$\hat{a}_x \cdot \hat{a}_\phi = -\sin\phi \qquad \hat{a}_y \cdot \hat{a}_\phi = \cos\phi \qquad \hat{a}_z \cdot \hat{a}_\phi = 0$$

$$\hat{a}_x \cdot \hat{a}_z = 0 \qquad \hat{a}_y \cdot \hat{a}_z = 0 \qquad \hat{a}_z \cdot \hat{a}_z = 1$$

Therefore, when we have a vector in Cartesian coordinate given by:

$$\vec{\mathbf{A}} = \mathbf{A}_x \hat{a}_x + \mathbf{A}_y \hat{a}_y + \mathbf{A}_z \hat{a}_z$$

Then this vector is transformed to cylindrical coordinate as follows:

$$\vec{\mathbf{A}}_\rho = \vec{\mathbf{A}} \cdot \hat{a}_\rho = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_\rho) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_\rho) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_\rho)$$

$$\vec{\mathbf{A}}_\phi = \vec{\mathbf{A}} \cdot \hat{a}_\phi = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_\phi)$$

$$\vec{\mathbf{A}}_z = \vec{\mathbf{A}} \cdot \hat{a}_z = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_z) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_z) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_z)$$

In matrix notation, we can write the transformation of vector (A) from $(\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z)$ to $(\mathbf{A}_\rho, \mathbf{A}_\phi, \mathbf{A}_z)$ as:

$$\begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \hat{a}_x \cdot \hat{a}_\rho & \hat{a}_y \cdot \hat{a}_\rho & \hat{a}_z \cdot \hat{a}_\rho \\ \hat{a}_x \cdot \hat{a}_\phi & \hat{a}_y \cdot \hat{a}_\phi & \hat{a}_z \cdot \hat{a}_\phi \\ \hat{a}_x \cdot \hat{a}_z & \hat{a}_y \cdot \hat{a}_z & \hat{a}_z \cdot \hat{a}_z \end{bmatrix} \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix}$$

While when we have a vector (A) in cylindrical coordinate given by: $\vec{\mathbf{A}}(\rho,\phi,z) = \mathbf{A}_\rho \hat{a}_\rho + \mathbf{A}_\phi \hat{a}_\phi + \mathbf{A}_z \hat{a}_z$, then this vector can be transformed to Cartesian coordinate as:

$$\vec{\mathbf{A}}_x = \vec{\mathbf{A}} \cdot \hat{a}_x = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_x) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_x) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_x)$$

$$\vec{\mathbf{A}}_y = \vec{\mathbf{A}} \cdot \hat{a}_y = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_y) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_y) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_y)$$

$$\vec{\mathbf{A}}_z = \vec{\mathbf{A}} \cdot \hat{a}_z = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_z) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_z) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_z)$$

These equations in matrix notation can be written as:

$$\begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \hat{a}_\rho \cdot \hat{a}_x & \hat{a}_\phi \cdot \hat{a}_x & \hat{a}_z \cdot \hat{a}_x \\ \hat{a}_\rho \cdot \hat{a}_y & \hat{a}_\phi \cdot \hat{a}_y & \hat{a}_z \cdot \hat{a}_y \\ \hat{a}_\rho \cdot \hat{a}_z & \hat{a}_\phi \cdot \hat{a}_z & \hat{a}_z \cdot \hat{a}_z \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix}$$

Therefore, the vector (A) can be transformed from $(\mathbf{A}_\rho, \mathbf{A}_\phi, \mathbf{A}_z)$ to $(\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z)$ by the matrix:

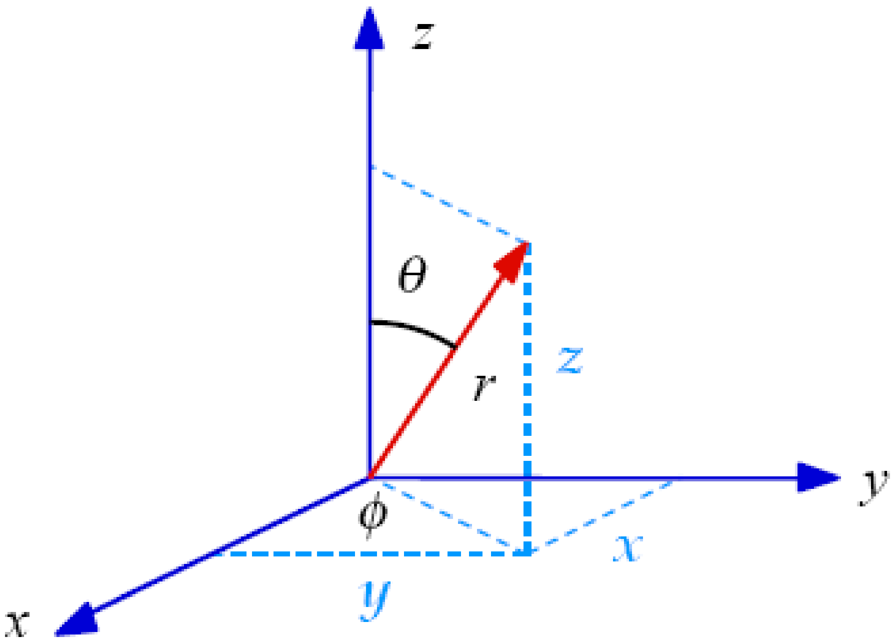
$$\begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix}$$

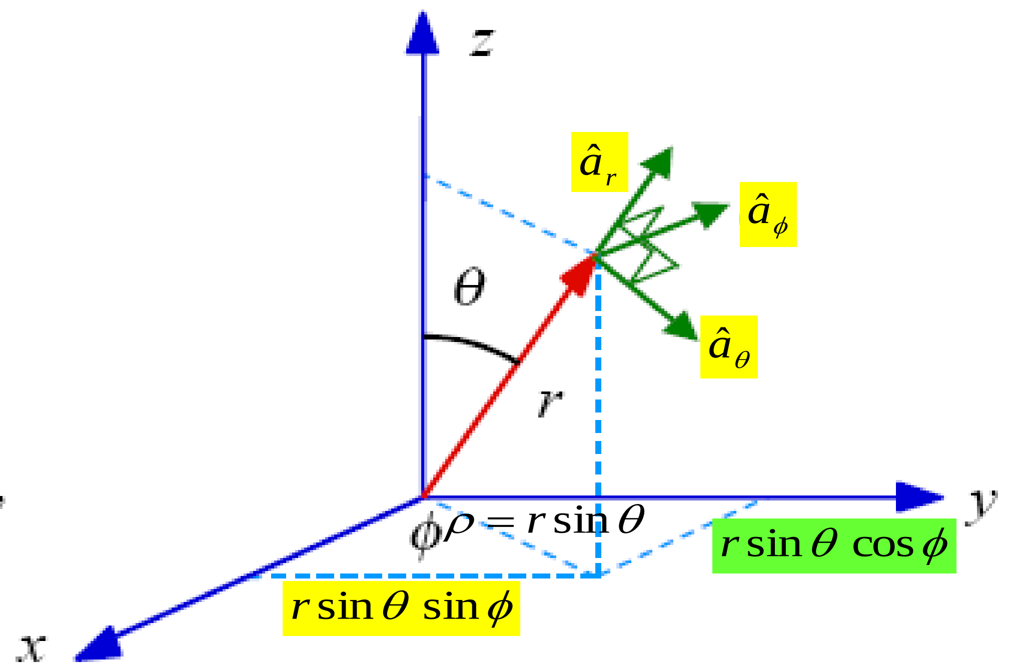
2-1-2: Cartesian to Spherical Transformation:

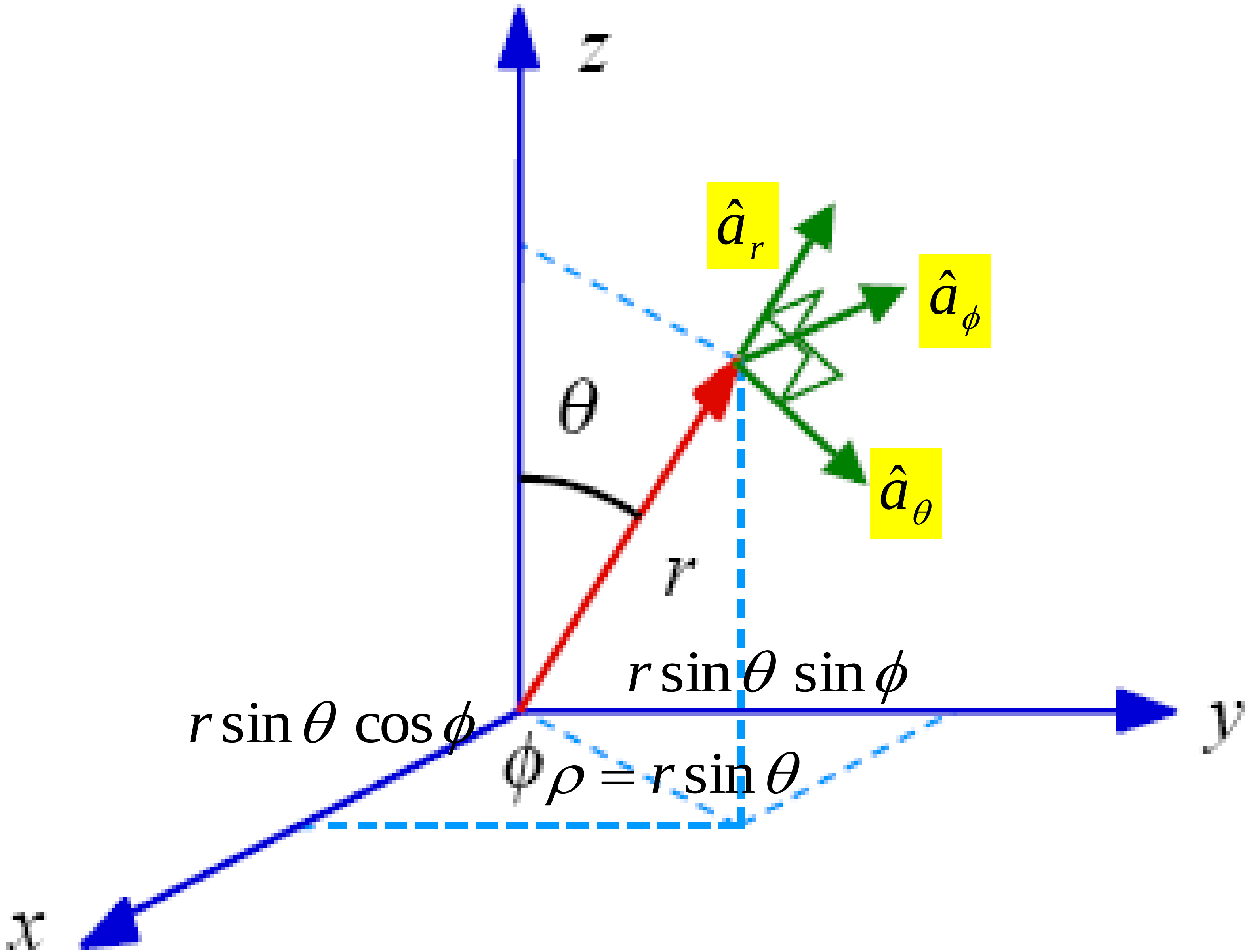
Point (P) in the figure has Cartesian coordinate (x, y, z) **and spherical coordinate** (r, θ, ϕ) **. The relation between the coordinates can be obtained as follows:**

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$







z
$\hat{a}_r$
$\hat{a}_\phi$
θ
$\hat{a}_\theta$
r
$r\sin\theta\,\sin\phi$
$r\sin\theta\,\cos\phi$
y
ϕ
$\rho = r\sin\theta$
x

The unit vector of Cartesian and spherical coordinates are related to each other through the following relations:

$$\hat{a}_x \cdot \hat{a}_r = \sin\theta\cos\phi$$
$$\hat{a}_x \cdot \hat{a}_\theta = \cos\theta\,\cos\phi$$
$$\hat{a}_x \cdot \hat{a}_\phi = -\sin\phi$$

$$\hat{a}_y \cdot \hat{a}_r = \sin\theta\,\sin\phi$$
$$\hat{a}_y \cdot \hat{a}_\theta = \cos\theta\,\sin\phi$$
$$\hat{a}_y \cdot \hat{a}_\phi = \cos\phi$$

$$\hat{a}_z \cdot \hat{a}_r = \cos\theta$$
$$\hat{a}_z \cdot \hat{a}_\theta = -\sin\theta$$
$$\hat{a}_z \cdot \hat{a}_\phi = 0$$

Therefore, when we have a vector in Cartesian coordinate given by:

$$\vec{\mathbf{A}} = \mathbf{A}_x\,\hat{a}_x + \mathbf{A}_y\,\hat{a}_y + \mathbf{A}_z\,\hat{a}_z$$

Then this vector is transformed to spherical coordinate as follows:

$$\vec{\mathbf{A}}_r = \vec{\mathbf{A}} \cdot \hat{a}_r = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_r) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_r) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_r)$$
$$\vec{\mathbf{A}}_\theta = \vec{\mathbf{A}} \cdot \hat{a}_\theta = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_\theta) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_\theta) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_\theta)$$
$$\vec{\mathbf{A}}_\phi = \vec{\mathbf{A}} \cdot \hat{a}_\phi = \vec{\mathbf{A}}_x(\hat{a}_x \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_y(\hat{a}_y \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_z(\hat{a}_z \cdot \hat{a}_\phi)$$

In matrix notation, we can write the transformation of vector (A) from $(\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z)$ **to** $(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_\phi)$ **as:**

$$\begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \hat{a}_x \cdot \hat{a}_r & \hat{a}_y \cdot \hat{a}_r & \hat{a}_z \cdot \hat{a}_r \\ \hat{a}_x \cdot \hat{a}_\theta & \hat{a}_y \cdot \hat{a}_\theta & \hat{a}_z \cdot \hat{a}_\theta \\ \hat{a}_x \cdot \hat{a}_\phi & \hat{a}_y \cdot \hat{a}_\phi & \hat{a}_z \cdot \hat{a}_\phi \end{bmatrix} \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix}$$

While when we have a vector (A) in spherical coordinate given by:

$$\vec{\mathbf{A}}(r,\theta,\phi) = \mathbf{A}_r\,\hat{a}_r + \mathbf{A}_\theta\,\hat{a}_\theta + \mathbf{A}_\phi\,\hat{a}_\phi$$

Then this vector can be transformed to Cartesian coordinate as:

$$\vec{\mathbf{A}}_x = \vec{\mathbf{A}}\cdot\hat{a}_x = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_x) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_x) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_x)$$

$$\vec{\mathbf{A}}_y = \vec{\mathbf{A}}\cdot\hat{a}_y = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_y) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_y) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_y)$$

$$\vec{\mathbf{A}}_z = \vec{\mathbf{A}}\cdot\hat{a}_z = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_z) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_z) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_z)$$

These equations in matrix notation can be written as:

$$\begin{bmatrix}\mathbf{A}_x\\ \mathbf{A}_y\\ \mathbf{A}_z\end{bmatrix} = \begin{bmatrix}\hat{a}_r\cdot\hat{a}_x & \hat{a}_\theta\cdot\hat{a}_x & \hat{a}_\phi\cdot\hat{a}_x\\ \hat{a}_r\cdot\hat{a}_y & \hat{a}_\theta\cdot\hat{a}_y & \hat{a}_\phi\cdot\hat{a}_y\\ \hat{a}_r\cdot\hat{a}_z & \hat{a}_\theta\cdot\hat{a}_z & \hat{a}_\phi\cdot\hat{a}_z\end{bmatrix}\begin{bmatrix}\mathbf{A}_r\\ \mathbf{A}_\theta\\ \mathbf{A}_\phi\end{bmatrix}$$

Therefore, the vector (A) can be transformed from $(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_\phi)$ **to** $(\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z)$ **by this matrix:**

$$\begin{bmatrix}\mathbf{A}_x\\ \mathbf{A}_y\\ \mathbf{A}_z\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\mathbf{A}_r\\ \mathbf{A}_\theta\\ \mathbf{A}_\phi\end{bmatrix}$$

2-1-3: Cylindrical to Spherical Transformation:

Point (P) in the figure has cylindrical coordinate (ρ, ϕ, z) **and spherical coordinate** (r, θ, ϕ) **. The relation between the coordinates can be obtained as follows:**

$$x = r \sin\theta \cos\phi \quad \text{and} \quad x = \rho \cos\phi$$

$$y = r \sin\theta \sin\phi \quad \text{and} \quad y = \rho \sin\phi$$

$$z = r \cos\theta \quad \text{and} \quad z = z$$

$$\rho = r \sin\theta$$

$$z = r \cos\theta$$

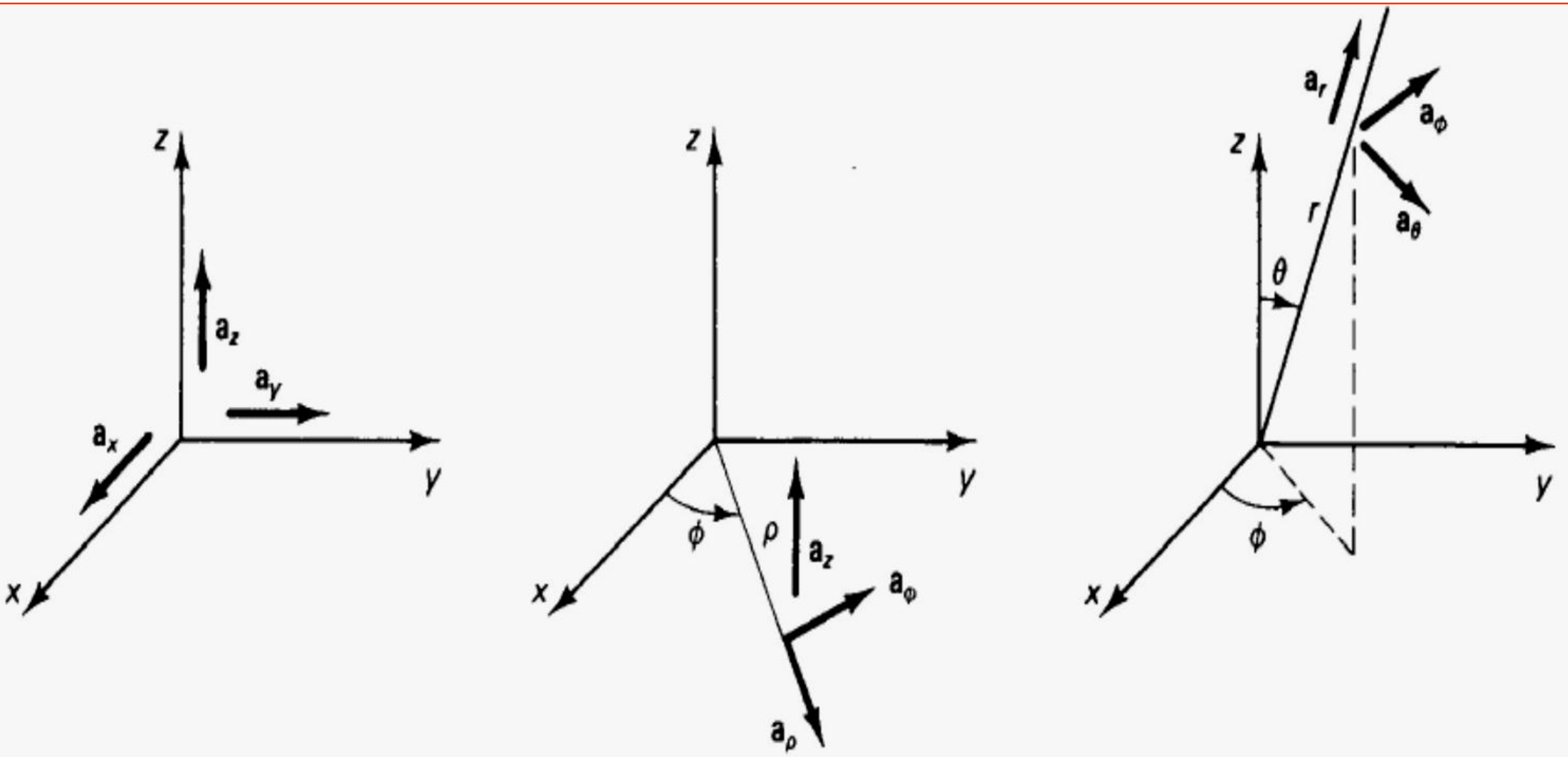
$$\phi = \phi$$

$$r = \sqrt{\rho^2 + z^2}$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$



Cartesian Cylindrical Spherical

The unit vector of Cylindrical and spherical coordinates are related to each other through the following relations:

$$\hat{a}_\rho \cdot \hat{a}_r = \sin\theta \qquad \hat{a}_\rho \cdot \hat{a}_\theta = \cos\theta \qquad \hat{a}_\rho \cdot \hat{a}_\phi = 0$$

$$\hat{a}_\phi \cdot \hat{a}_r = 0 \qquad \hat{a}_\phi \cdot \hat{a}_\theta = 0 \qquad \hat{a}_\phi \cdot \hat{a}_\phi = 1$$

$$\hat{a}_z \cdot \hat{a}_r = \cos\theta \qquad \hat{a}_z \cdot \hat{a}_\theta = -\sin\theta \qquad \hat{a}_z \cdot \hat{a}_\phi = 0$$

Therefore, when we have a vector in Cartesian coordinate given by:

$$\vec{\mathbf{A}} = \mathbf{A}_\rho \hat{a}_\rho + \mathbf{A}_\phi \hat{a}_\phi + \mathbf{A}_z \hat{a}_z$$

Then this vector is transformed to spherical coordinate as follows:

$$\vec{\mathbf{A}}_r = \vec{\mathbf{A}} \cdot \hat{a}_r = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_r) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_r) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_r)$$

$$\vec{\mathbf{A}}_\theta = \vec{\mathbf{A}} \cdot \hat{a}_\theta = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_\theta) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_\theta) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_\theta)$$

$$\vec{\mathbf{A}}_\phi = \vec{\mathbf{A}} \cdot \hat{a}_\phi = \vec{\mathbf{A}}_\rho (\hat{a}_\rho \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_\phi (\hat{a}_\phi \cdot \hat{a}_\phi) + \vec{\mathbf{A}}_z (\hat{a}_z \cdot \hat{a}_\phi)$$

In matrix notation, we can write the transformation of vector (A) from $(\mathbf{A}_\rho, \mathbf{A}_\phi, \mathbf{A}_z)$ **to** $(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_\phi)$ **as:**

$$\begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \hat{a}_\rho \cdot \hat{a}_r & \hat{a}_\phi \cdot \hat{a}_r & \hat{a}_z \cdot \hat{a}_r \\ \hat{a}_\rho \cdot \hat{a}_\theta & \hat{a}_\phi \cdot \hat{a}_\theta & \hat{a}_z \cdot \hat{a}_\theta \\ \hat{a}_\rho \cdot \hat{a}_\phi & \hat{a}_\phi \cdot \hat{a}_\phi & \hat{a}_z \cdot \hat{a}_\phi \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix}$$

While when we have a vector (A) in spherical coordinate given by:

$$\vec{\mathbf{A}}(r,\theta,\phi) = \mathbf{A}_r\,\hat{a}_r + \mathbf{A}_\theta\,\hat{a}_\theta + \mathbf{A}_\phi\,\hat{a}_\phi$$

Then this vector can be transformed to Cartesian coordinate as:

$$\vec{\mathbf{A}}_\rho = \vec{\mathbf{A}}\cdot\hat{a}_\rho = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_\rho) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_\rho) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_\rho)$$

$$\vec{\mathbf{A}}_\phi = \vec{\mathbf{A}}\cdot\hat{a}_\phi = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_\phi) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_\phi) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_\phi)$$

$$\vec{\mathbf{A}}_z = \vec{\mathbf{A}}\cdot\hat{a}_z = \vec{\mathbf{A}}_r(\hat{a}_r\cdot\hat{a}_z) + \vec{\mathbf{A}}_\theta(\hat{a}_\theta\cdot\hat{a}_z) + \vec{\mathbf{A}}_\phi(\hat{a}_\phi\cdot\hat{a}_z)$$

These equations in matrix notation can be written as:

$$\begin{bmatrix}\mathbf{A}_\rho\\ \mathbf{A}_\phi\\ \mathbf{A}_z\end{bmatrix} = \begin{bmatrix}\hat{a}_r\cdot\hat{a}_\rho & \hat{a}_\theta\cdot\hat{a}_\rho & \hat{a}_\phi\cdot\hat{a}_\rho\\ \hat{a}_r\cdot\hat{a}_\phi & \hat{a}_\theta\cdot\hat{a}_\phi & \hat{a}_\phi\cdot\hat{a}_\phi\\ \hat{a}_r\cdot\hat{a}_z & \hat{a}_\theta\cdot\hat{a}_z & \hat{a}_\phi\cdot\hat{a}_z\end{bmatrix}\begin{bmatrix}\mathbf{A}_r\\ \mathbf{A}_\theta\\ \mathbf{A}_\phi\end{bmatrix}$$

Therefore, the vector (A) can be transformed from $(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_\phi)$ to $(\mathbf{A}_\rho, \mathbf{A}_\phi, \mathbf{A}_z)$ by this matrix:

$$\begin{bmatrix}\mathbf{A}_\rho\\ \mathbf{A}_\phi\\ \mathbf{A}_z\end{bmatrix} = \begin{bmatrix}\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\mathbf{A}_r\\ \mathbf{A}_\theta\\ \mathbf{A}_\phi\end{bmatrix}$$

Example(1): Let , $\vec{A} = \rho\cos\phi\,\hat{a}_\rho + \rho z^2 \sin\phi\,\hat{a}_z$ then :

(a). Transform $\vec{A}$ into Cartesian coordinate and calculate its magnitude at point (3,-4,0).

(b). Transform $\vec{A}$ into spherical coordinate and calculate its magnitude at point (3,-4,0).

(a). since the vector is in cylindrical coordinate, and we have the following matrix to convert it to Cartesian coordinate:

$$\begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ \rho z^2 \sin\phi \end{bmatrix}$$

$$\mathbf{A}_x = \rho\cos^2\phi + 0 + 0$$
$$\mathbf{A}_y = \rho\,\sin\phi\,\cos\phi + 0 + 0$$
$$\mathbf{A}_z = 0 + 0 + \rho\,z^2 \sin\phi$$

at point (3,-4,0), we have :

$$\rho = \sqrt{x^2 + y^2} = \sqrt{9+16} = 5$$
$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1} - \frac{4}{3} = -53.13$$
$$z = 0$$

$$\therefore \quad \mathbf{A}_x = 5\cos^2(-53.13) = 1.8 \qquad \mathbf{A}_y = 5\sin(-53.13)\cos(-53.13) = -2.3 \qquad \mathbf{A}_z = 5\times 0\times\sin(-53.13) = 0$$

Therefore, the vector in Cartesian coordinate is:

$$\vec{\mathbf{A}} = \rho\,\cos^2\phi\,\hat{a}_x + \rho\cos\phi\sin\phi\,\hat{a}_y + \rho z^2 \sin\phi\,\hat{a}_z \Rightarrow \quad \vec{\mathbf{A}} = 1.8\,\hat{a}_x - 2.3\,\hat{a}_y + 0$$

(b). since the vector is in cylindrical coordinate, and we have the following matrix to convert it to spherical coordinate:

$$\begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}_\rho \\ \mathbf{A}_\phi \\ \mathbf{A}_z \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ \rho z^2 \sin\phi \end{bmatrix}$$

$$\mathbf{A}_r = \rho\sin\theta\cos\phi + \rho z^2 \cos\theta\sin\phi \qquad \mathbf{A}_\theta = \rho\cos\theta\cos\phi - \rho z^2\sin\theta\sin\phi \qquad \mathbf{A}_\phi = 0+0+0=0$$

The spherical point of (3, -4, 0) is given by :

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{9+16+0} = 5$$

$$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{0}{5} = 90^\circ$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1} -\frac{4}{3} = -53.13$$

Substituting these value in the above equations we get:

$$\mathbf{A}_r = 5\sin 90\cos(-53.13) + 5\times 0\times\cos 90\sin(-53.13) = 3.0$$

$$\mathbf{A}_\theta = 5\cos 90\cos(-53.13) - 5\times 0\times\sin 90\sin(-53.13) = 0$$

$$\mathbf{A}_\phi = 0$$

Therefore the final form of the vector in spherical coordinate is given by:

$$\vec{\mathbf{A}} = (\rho\sin\theta\cos\phi + \rho z^2\cos\theta\sin\phi)\,\hat{a}_r + (\rho\cos\theta\cos\phi - \rho z^2\sin\theta\sin\phi)\,\hat{a}_\theta + 0\,\hat{a}_\phi$$

$$\vec{\mathbf{A}} = 3.0\,\hat{a}_r + 0\,\hat{a}_\theta + 0\,\hat{a}_\phi \quad \Rightarrow \quad \vec{\mathbf{A}} = 3.0\,\hat{a}_r$$

Home Work

Q1/ Express the vector $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ **in spherical and cylindrical coordinates.**

Q2/Transform the vector field $\vec{F} = 2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta$ **into Cartesian coordinates?**

Q3/ Express the vector $\vec{A} = (x+y)\hat{a}_x + (y-x)\hat{a}_y + z\hat{a}_z$ **in spherical coordinate?**

Q_4/Transform the following vectors to cylindrical and spherical coordinates :

a. $\vec{A} = (x+y)\hat{a}_y$, **and** , **b.** $\vec{B} = (y^2 - x^2)\hat{a}_x + xyz\,\hat{a}_y + (x^2 - z^2)\hat{a}_z$

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Q_5/ Transform the following vectors into spherical coordinates and then evaluate them at the indicated points:

(a). $\vec{A} = y^2\,\hat{a}_x + xz\,\hat{a}_y + 4\hat{a}_z \;\; at \;\; po\text{int} \;\; P(1,-1,2).$

(b). $\vec{B} = (x^2 + y^2 + z^2)\hat{a}_y - (x^2 + y^2)\hat{a}_z \;\; at \;\; po\text{int} \;\; P(-1,0,2)$

(c). $\vec{C} = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi + \cos\phi\,\sin\phi\,\hat{a}_z \;\; at \;\; po\text{int} \;\; P(2,\frac{\pi}{4},2)$

2-2:Line, Surface and Volume Integral:

The familiar concept of integration will now be extended to cases when the integral involves a vector. By a line we mean the path along a curve in space.

The line integral $\int_L \vec{\mathbf{A}} \cdot \vec{dl}$ **is the integral of the tangential component of** $\vec{\mathbf{A}}$ *along curve* $\mathbf{L}$ $\int_L \vec{\mathbf{A}} \cdot \vec{dl} = \int_a^b |\mathbf{A}| \cos\theta \, dl$

, when the path of integration is a closed path such as (abca) as shown in figure, this integration becomes: $\oint_L \vec{\mathbf{A}} \cdot \vec{dl} = \oint_L |\mathbf{A}| \cos\theta \, dl$ **which called the circulation of** $\vec{\mathbf{A}}$ *around* $\mathbf{L}$

$$\overrightarrow{dl} = dx\ \hat{a}_x + dy\ \hat{a}_y + dz\ \hat{a}_z \quad ----Cartesian\ Coordinate$$

$$\overrightarrow{dl} = d\rho\ \hat{a}_\rho + \rho\, d\phi\ \hat{a}_\phi + dz\ \hat{a}_z ----Cylindrical\ Coordinate$$

$$\overrightarrow{dl} = dr\ \hat{a}_r + r\, d\theta\ \hat{a}_\theta + r\sin\theta\ d\phi\ \hat{a}_\phi \ --Spherical\ Coordinate$$

$$W = \int \vec{\mathbf{F}} \cdot \overrightarrow{dl}$$

$$I = \oint_C \vec{\mathbf{H}} \cdot \overrightarrow{dl}$$

· **The surface integral or the flux of a given vector** $\vec{\mathbf{A}}$ **through a smooth surface (**S**) is given by:**

$$\psi = \int_S \vec{\mathbf{A}} \cdot \overrightarrow{ds} = \int_S \vec{\mathbf{A}} \cdot \hat{a}_n\, ds = \int_S |\mathbf{A}| \cos\theta\ ds$$

Where, $\hat{a}_n$ is the unit vector normal to (S), and (ds) in all three coordinates are given as:

$$\overrightarrow{ds} = dy\,dz\ \hat{a}_x + dx\,dz\ \hat{a}_y + dx\,dy\ \hat{a}_z \qquad ----Cartesian\ Coordinate$$

$$\overrightarrow{ds} = \rho\ dz\,d\phi\ \hat{a}_\rho + d\rho\,dz\ \hat{a}_\phi + \rho\,d\rho\,d\phi\ \hat{a}_z ----Cylindrical\ Coordinate$$

$$\overrightarrow{ds} = r^2 \sin\theta\ d\theta\,d\phi\ \hat{a}_r + r\sin\theta\ dr\,d\phi\ \hat{a}_\theta + r\,dr\ d\theta\ \hat{a}_\phi \ --Spherical\ Coordinate$$

for a closed surface we define a volume, and the previous integral becomes. $\psi = \oint \vec{\mathbf{A}} \cdot \overrightarrow{ds}$

A closed path defines an open surface whereas a closed surface defines a volume, and the volume integral is given by:

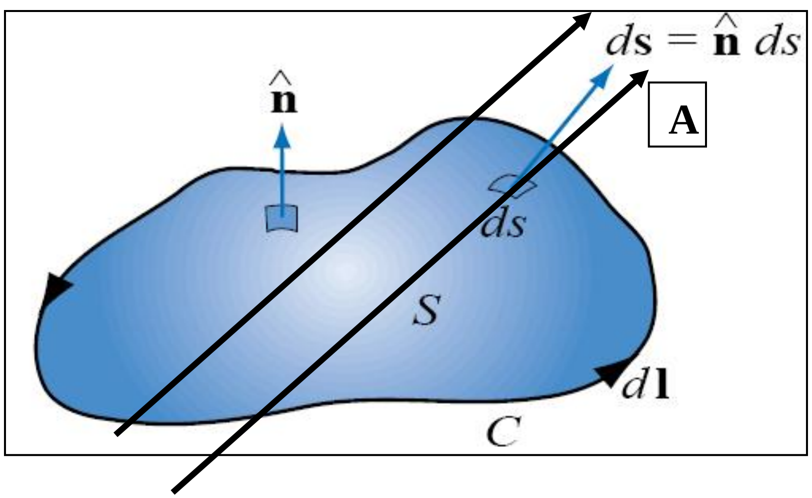
$$\psi = \int_V \rho_v\, dV \qquad , \rho_v\ is\ a\ scalar\ quantity$$

Where, (dv) in all three coordinates are given as:

$$dv = dx\,dy\,dz \qquad ----Cartesian\ Coordinate$$

$$dv = \rho\,d\rho\,d\phi\,dz ----Cylindrical\ Coordinate$$

$$dv = r^2 \sin\theta\ dr\ d\theta\ d\phi - Spherical\ coordinate$$



Example(2):

(a). find the length of each of the curves: $\rho = 3, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{2} \,, \quad z = cons\tan t.$

(b). Calculate the area of the surfaces area defined by: $z = 1, 1 < \rho < 3 \,, 0 < \phi < \frac{\pi}{4}$

(c). determine the volume of the region defined by: $1 < r < 3 \,, \quad \frac{\pi}{2} < \theta < \frac{2\pi}{3} \,, \quad \frac{\pi}{6} < \phi < \frac{\pi}{2}$

Solution:

(a).

$$dl = \rho \, d\phi \qquad \therefore \quad \int dl = \int_{\pi/4}^{\pi/2} \rho \, d\phi = 3\,(\pi/2 - \pi/4) \quad \Rightarrow \quad l = \frac{3}{4}\pi \ unit \ \ length$$

(b).

$$ds = \rho \ d\rho \ d\phi \qquad \therefore \quad \int ds = \int_1^3 \int_0^{\pi/4} \rho \ d\rho \ d\phi = \frac{1}{2}\,(\rho^2)\Big|_1^3 \ (\phi)\Big|_0^{\pi/4} = 4 \times \frac{\pi}{4} \quad \Rightarrow \quad S = \pi \ unit \ of \ \ area$$

(c).

$$dv = r^2 \sin\theta \ dr \ d\theta \ d\phi \quad \therefore \int dv = \int_1^3 \int_{\pi/2}^{2\pi/3} \int_{\pi/6}^{\pi/2} r^2 \sin\theta \ dr \ d\theta \ d\phi = \frac{r^3}{3}\Big|_1^3 \ (-\cos\theta)\Big|_{\pi/2}^{2\pi/3} \ (\phi)\Big|_{\pi/6}^{\pi/2}$$

$$\therefore \ v = \frac{1}{3}(27 - 1) \times (\cos\pi/2 - \cos 2\pi/3) \times (\pi/2 - \pi/6)$$

$$v = \frac{1}{3}(26) \times (0 + 0.5) \times (\frac{\pi}{3}) \qquad \Rightarrow \ v = \frac{13}{9}\,\pi \quad unit \ of \ \ volume$$

Home Work

Q_1/ Using the differential length dl, to find the length of each of the following curves

a- $\rho = 3, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{2}, \quad z = constant.$

b- $r = 1, \ \theta = 30^\circ \ , 0 \le \phi \le 60^\circ$

c- $r = 4, \ 30^\circ \le \theta \le 90^\circ, \ \phi = constant$

Q_2/ Calculate the area of the following surfaces using the differential surface ar ds

a- $\rho = 2, \ \frac{\pi}{3} \le \phi \le \frac{\pi}{2} \ , 0 < z < 5$

b- $z = 1, 1 < \rho < 3, \ 0 < \phi < \frac{\pi}{4}$

c- $0 < r < 4, \ 60^\circ < \theta < 90^\circ, \ \phi = Constant$

Q_3/ Use the differential volume dv to determine the volume of the following regions

a- $0 < x < 1 \ , 1 < y < 2 \ , \ -3 < z < 3$

b- $2 < \rho < 5, \ \frac{\pi}{3} < \phi < \pi, \ -1 < z < 4$

c- $1 < r < 3 \ , \ \frac{\pi}{2} < \theta < \frac{2\pi}{3} \ , \ \frac{\pi}{6} < \phi < \frac{\pi}{2}$

Q_4/ Using spherical coordinates to express the differential volume integrate, to obtain the volume defined by : $1 \le r \le 2 \ m \ , \quad 0 \le \theta \le \frac{\pi}{2} \quad , \quad 0 \le \phi \le \frac{\pi}{2}$

Q_5/ Find the area of a cylindrical surface described by :

$$\rho = 5 \ , \ \ 30 \le \phi \le 60 \quad and \quad 0 \le z \le 3$$