2-3: Del Operator:

When dealing with a scalar quantity whose magnitude depends on a single variable, such as the temperature (T) as a function of height (Z), the rate of change of (T) with height can be described by $\frac{dT}{dZ}$

However, if (T) is also a function of $x, y \text{ and } z$ **in a Cartesian coordinate system, its space rate of change becomes more difficult to describe it. The differential change in (T) along** $x, y \text{ and } z$ **can be described in terms of the partial derivatives of (T) with respect to the three coordinate systems,**

$$\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$

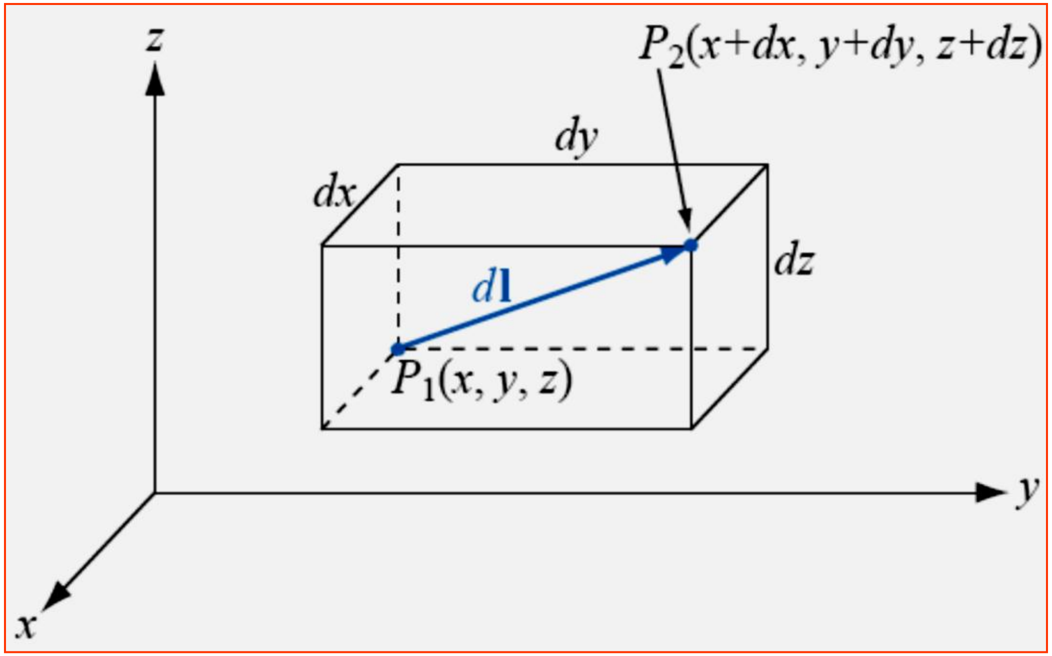
Furthermore, many of the quantities we deal with in EMF, are vectors and therefore, both magnitude and direction may vary with spatial position. In vector calculus, we use three fundamental operators to describe the differential spatial variation of Scalar and Vectors: these are the Gradient, Divergence, and Curl

2-3-1: Gradient Operator:

The gradient operator applies to scalar fields and is defined as a vector represents both the magnitude and the direction of the maximum space rate of increase of the scalar field.

From this figure, if $T_1(x,y,z)$ **is the temperature at point** $P_1(x,y,z)$ **and** $T_2(x+dx,y+dy,z+dz)$ **is the temperature at a nearby point** $P_2(x+dx,y+dy,z+dz)$ **. Then the differential distances** $dx, dy\ and\ dz$ **are the components of the differential distance vector** $\overrightarrow{dl}$ **, that is:**

$$\overrightarrow{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z \quad --------(1)$$



The mathematical expression for the gradient can be obtained by evaluating the difference in the field dT between point P_1 and P_2. From differential calculus, the differential temperature $dT = T_2 - T_1$ and is given by:

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz ----------(2)$$

Since, $dx = \overrightarrow{dl} \cdot \hat{a}_x \quad , \quad dy = \overrightarrow{dl} \cdot \hat{a}_y \quad and \quad dz = \overrightarrow{dl} \cdot \hat{a}_z$, then eq.(2) can be written as:

$$dT = \left(\frac{\partial T}{\partial x}\right) \hat{a}_x \cdot \overrightarrow{dl} + \left(\frac{\partial T}{\partial y}\right) \hat{a}_y \cdot \overrightarrow{dl} + \left(\frac{\partial T}{\partial z}\right) \hat{a}_z \cdot \overrightarrow{dl}$$

$$dT = \left[\left(\frac{\partial T}{\partial x}\right) \hat{a}_x + \left(\frac{\partial T}{\partial y}\right) \hat{a}_y + \left(\frac{\partial T}{\partial z}\right) \hat{a}_z \right] \cdot \overrightarrow{dl} -------(3)$$

The vector inside the square bracket in eq.(3) defines the change in temperature dT corresponding to a vector change $\overrightarrow{dl}$ in position. This vector is called the Gradient of (T) or grad, that is:

$$\vec{\nabla} T = grad\, T = \frac{\partial T}{\partial x} \hat{a}_x + \frac{\partial T}{\partial y} \hat{a}_y + \frac{\partial T}{\partial z} \hat{a}_z -------(4)$$

$$then \;:\; dT = \vec{\nabla} T \cdot \overrightarrow{dl} ----------------(5)$$

The symbol (∇) is called del or gradient operators and is expressed in Cartesian, Cylindrical and spherical coordinate systems as follows respectively:

$$\vec{\nabla}_{x,y,z} = \frac{\partial}{\partial x}\,\hat{a}_x + \frac{\partial}{\partial y}\,\hat{a}_y + \frac{\partial}{\partial z}\,\hat{a}_z \qquad ------Cartesian$$

$$\vec{\nabla}_{\rho,\phi,z} = \frac{\partial}{\partial \rho}\,\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\,\hat{a}_\phi + \frac{\partial}{\partial z}\,\hat{a}_z \qquad ------Cylindrical$$

$$\vec{\nabla}_{r,\theta,\phi} = \frac{\partial}{\partial r}\,\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\,\hat{a}_\phi \qquad ------Sphercal$$

Notes:

1.The gradient operator (∇ has no physical meaning by itself, it attains a physical meaning once it operate on a scalar physical quantity.

2. The result of the operation of (∇) is a vector whose magnitude is equal to the maximum rate of change of the physical quantity per unit distance and whose direction is along the direction of maximum increase of the scalar field.

Properties of (∇) operator:

(1). $\vec{\nabla}(V+U) = \vec{\nabla}V + \vec{\nabla}U$

(2). $\vec{\nabla}(VU) = V\vec{\nabla}U + U\vec{\nabla}V$

(3). $\vec{\nabla}\left(\frac{U}{V}\right) = \frac{V\vec{\nabla}U - U\vec{\nabla}V}{V^2}$

(4). $\vec{\nabla}V^n = nV^{n-1}\vec{\nabla}V$ where, U and V are scalar quantity and (n) is an integer.

Example(1): Find the gradient of the following scalar functions:

a. $U = 4xz^2 + 3yz$ **b.** $W = 2\rho(z^2+1)\cos\phi$ **c.** $V = r^2\cos\theta\cos\phi$

Solution:

(a). $$\vec{\nabla}U = (\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z)(4xz^2+3yz) = 4z^2\hat{a}_x + 3z\hat{a}_y + (8xz+3y)\hat{a}_z$$

(b). $$\vec{\nabla}W = (\frac{\partial}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z)(2\rho(z^2+1)\cos\phi) = 2(z^2+1)\cos\phi\,\hat{a}_\rho - 2(z^2+1)\sin\phi\,\hat{a}_\phi + 4\rho z\cos\phi\,\hat{a}_z$$

.

(c). $$\vec{\nabla}V = (\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{a}_\phi)(r^2\cos\theta\cos\phi) = 2r\cos\theta\cos\phi\,\hat{a}_r - r\sin\theta\cos\phi\,\hat{a}_\theta - r\cot\theta\sin\phi\,\hat{a}_\phi$$

Home Work

Q_1/ Prove that : $\nabla\cdot(V\vec{A}) = V\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla V$ **, where V is a scalar field and** $\vec{A}$ **is a vector field ?**

Q_2/ prove that : $\nabla\times(V\vec{A}) = V\,\nabla\times\vec{A} + \nabla V\times\vec{A}$**, where V is a scalar field and** $\vec{A}$ **is a vector field**

Q_3/ The heat flow vector $\vec{H} = k\,\nabla T$**, where T is the temperature and k is the thermal conductivity. Show that where** $T = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}$ **, then** $\nabla\cdot\vec{H} = 0$ **or** $\vec{H}$ **is solenoid.**

Q4/ Find the gradient of each of the following scalar functions and evaluate it at the given point:

a. $V = 4V_\circ\cos\left(\frac{\pi y}{3}\right)\sin\left(\frac{2\pi z}{3}\right) \quad at \quad (2,3,1)$

b. $V = V_\circ\, e^{-2\rho}\, Sin3\phi \quad at \quad (1, 90^\circ, 3)$

c. $V = V_\circ\frac{a}{r}\;Cos2\theta \quad at \quad (2a, 0^\circ, \pi)$

Q_5/ Knowing that $\vec{D} = D_x\,\hat{a}_x\,,\ and\ that\ \nabla\cdot\vec{D} = x + y$**, then find the general solution of** $\vec{D}$**?**

2-3-2: Divergence of a Vector and Divergence Theorem:

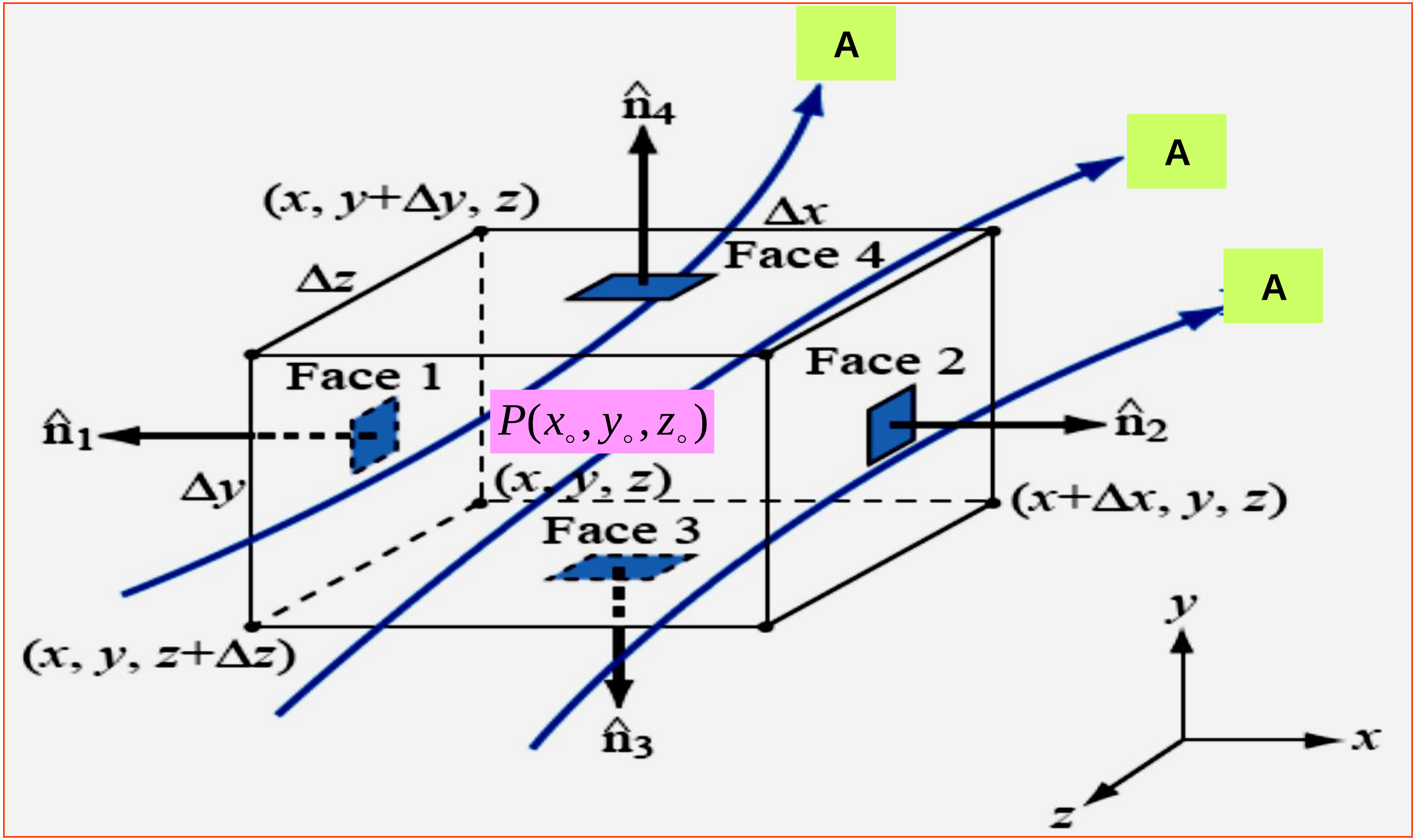
The divergence theorem, states that the total outward flux of a vector field ($\vec{\mathbf{A}}$) through the closed surface **(S)** is the same as the volume integral of the divergence of ($\vec{\mathbf{A}}$)

$$\oint_S \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{ds}} = \int \vec{\nabla} \cdot \vec{\mathbf{A}} \; dv -------(1-a)$$

$$div\,\vec{\mathbf{A}} = \vec{\nabla} \cdot \vec{\mathbf{A}} = \lim_{\Delta v \to 0} it \frac{\oint_S \vec{\mathbf{A}} \cdot \overrightarrow{ds}}{\Delta v} \quad -----(1-b)$$

Definition: The divergence of the vector field $\vec{\mathbf{A}}$ at a given point is a measure of how much the field diverges or emanates from that point.

Suppose we wish to evaluate the divergence of ($\vec{A}$) at point $P(x_\circ, y_\circ, z_\circ)$



we let the point be enclosed by a differential volume as shown in the figure, then the surface integral can be obtained as:

$$\oint_S \vec{\mathbf{A}}\cdot\overrightarrow{ds} = \oint_{Face\,1} \vec{\mathbf{A}}\cdot\overrightarrow{ds} + \oint_{Face\,2} \vec{\mathbf{A}}\cdot\overrightarrow{ds} + \oint_{Face\,3} \vec{\mathbf{A}}\cdot\overrightarrow{ds} + \oint_{Face\,4} \vec{\mathbf{A}}\cdot\overrightarrow{ds} + \oint_{Face\,5} \vec{\mathbf{A}}\cdot\overrightarrow{ds} + \oint_{Face\,6} \vec{\mathbf{A}}\cdot\overrightarrow{ds} \quad ------(2)$$

A three dimensional Taylor series expression of $\mathbf{A}_x \; about \;\; P$ **is:**

$$\mathbf{A}(x,y,z) = \mathbf{A}(x_\circ, y_\circ, z_\circ) + (x-x_\circ)\frac{\partial \mathbf{A}_x}{\partial x}\Big|_P + (y-y_\circ)\frac{\partial \mathbf{A}_x}{\partial y}\Big|_P + (z-z_\circ)\frac{\partial \mathbf{A}_x}{\partial z}\Big|_P + Higher\;\; order\, term --(3)$$

For the front side (face 2), $x = x_\circ + \dfrac{dx}{2} \;, \quad and \quad \overrightarrow{\mathbf{ds}} = dy\, dz \;\hat{a}_x \;,$ **then:**

$$\int_{face2} \vec{\mathbf{A}}\cdot\overrightarrow{\mathbf{ds}} = dy\, dz\left[\mathbf{A}_x(x_\circ, y_\circ, z_\circ) + \frac{dx}{2}\,\frac{\partial \mathbf{A}_x}{\partial x}\Big|_P\right] + higher\;\; order \;\; ----------(4)$$

For the back side (face 1), $x = x_\circ - \dfrac{dx}{2} \;, \quad and \quad \overrightarrow{\mathbf{ds}} = dy\, dz\,(-\hat{a}_x) \;,$ **then:**

$$\int_{face1} \vec{\mathbf{A}}\cdot\overrightarrow{\mathbf{ds}} = \; -dy\, dz\left[\mathbf{A}_x(x_\circ, y_\circ, z_\circ) - \frac{dx}{2}\,\frac{\partial \mathbf{A}_x}{\partial x}\Big|_P\right] + higher\;\; order \;\; ---------(5)$$

Hence,

$$\int_{face2} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} + \int_{face1} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} = dx\,dy\,dz\ \frac{\partial \mathbf{A}_x}{\partial x} + higher\ order ----------(6)$$

By taking the similar steps we can obtain the following expressions:

$$\int_{face3} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} + \int_{face4} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} = dx\,dy\,dz\ \frac{\partial \mathbf{A}_y}{\partial y} + higher\ order ----------(7)$$

$$\int_{face5} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} + \int_{face6} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s} = dx\,dy\,dz\ \frac{\partial \mathbf{A}_z}{\partial z} + higher\ order ----------(8)$$

Substituting eqs.(6), (7) and (8) into eq.(2) and noting that

$$\lim_{\Delta v \to o} \frac{\oint \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{d}s}}{\Delta v} = \left(\frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z} \right) \Big|_P -------------(9)$$

Because the higher order terms will vanishes as $\Delta v \to 0$ **. Thus, the divergence of** $\vec{\mathbf{A}}$ **at point** $P(x_\circ, y_\circ, z_\circ)$ **in a Cartesian coordinate system is given by:**

$$\vec{\nabla} \cdot \vec{\mathbf{A}} = \frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z}$$

while in cylindrical and spherical coordinate system is expressed as:

$$\vec{\nabla} \cdot \vec{\mathbf{A}} = \frac{1}{\rho} \frac{\partial (\rho \mathbf{A}_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial \mathbf{A}_\phi}{\partial \phi} + \frac{\partial \mathbf{A}_z}{\partial z} \quad ------------- \mathit{cylindrical\ coord.}$$

$$\vec{\nabla} \cdot \vec{\mathbf{A}} = \frac{1}{r^2} \frac{\partial (r^2 \mathbf{A}_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta \, \mathbf{A}_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \mathbf{A}_\phi}{\partial \phi} \quad ------- \mathit{spherical\ coord.}$$

The divergence of a vector field has the following properties:

(1). It is produce a scalar field **(2).** $\vec{\nabla} \cdot (\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \vec{\nabla} \cdot \vec{\mathbf{A}} + \vec{\nabla} \cdot \vec{\mathbf{B}}$

(3). $\vec{\nabla} \cdot (V\vec{\mathbf{A}}) = V\vec{\nabla} \cdot \vec{\mathbf{A}} + \vec{\mathbf{A}} \cdot \vec{\nabla} V \qquad \textit{where, } V \textit{ is a scalar quantity}$

Example(2): For the vector field $\vec{E} = 10\rho e^{-\rho}\,\hat{a}_\rho - 3z\,\hat{a}_z$

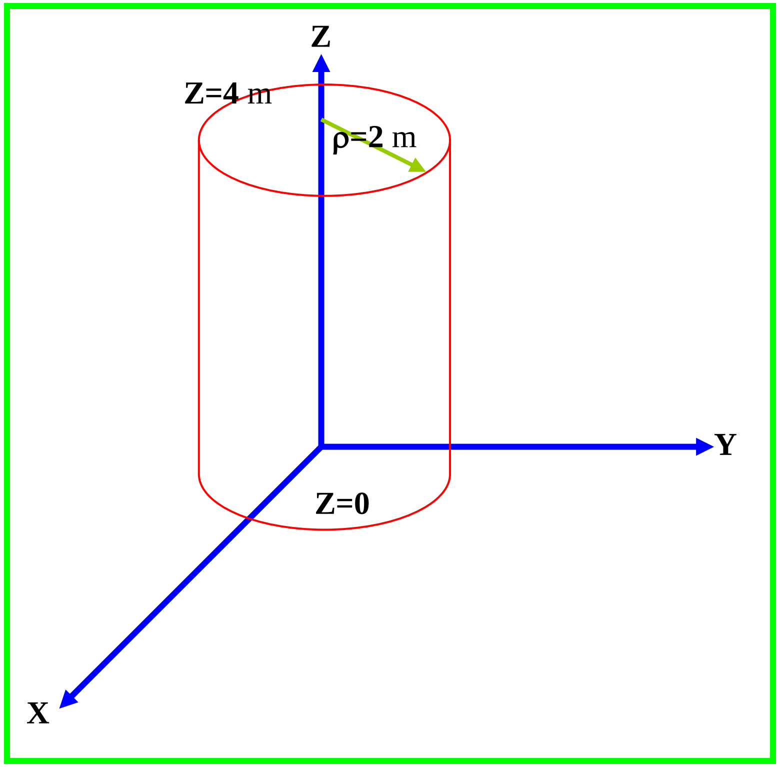
, verify the divergence theorem for the cylindrical region enclosed by :

$$\rho = 2\ m\ ,\quad z = 0 \quad and \quad z = 4\ m$$

Solution:

The divergence theorem mathematically states that : $\oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \int (\vec{\nabla} \cdot \vec{\mathbf{E}})\ dv$

There are three closed surfaces bounded by the defined region, they are:



$$ds_z\ \ at\ \ (z=0) = \rho\, d\rho\, d\phi\, (-\hat{a}_z)$$

$$ds_z\ \ at\ \ (z=4m) = \rho\, d\rho\, d\phi\, \hat{a}_z$$

$$ds_\rho\ \ at\ \ (\rho=2) = \rho\, dz\, d\phi\, \hat{a}_\rho$$

Therefore the right hand side of the equation of divergence theorem is:

$$\oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \oint_{S1} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} + \oint_{S2} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} + \oint_{S3} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}$$

$$\oint_{S1} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \int_0^{2\pi}\int_0^{2} (10\rho e^{-\rho}\hat{a}_\rho - 3z\hat{a}_z) \cdot \rho\, d\rho\, d\phi (-\hat{a}_z) = \int_0^{2\pi}\int_0^{2} 3z\ \rho\, d\rho\, d\phi = 3z\ \frac{1}{2}\rho^2 (2\pi) = 3 \times 0 \times 2 \times 2\pi = 0$$

$$\oint_{S2} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \int_0^{2\pi}\int_0^{2} (10\rho e^{-\rho}\hat{a}_\rho - 3z\hat{a}_z) \cdot \rho\, d\rho\, d\phi\, \hat{a}_z = \int_0^{2\pi}\int_0^{2} -3z\ \rho\, d\rho\, d\phi = -3z\ \frac{1}{2}\rho^2 (2\pi) = -3 \times 4 \times 2 \times 2\pi = -48\pi$$

$$\oint_{S2} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \int_0^{2\pi}\int_0^{2} (10\rho e^{-\rho}\hat{a}_\rho - 3z\hat{a}_z) \cdot \rho\, dz\, d\phi\, \hat{a}_\rho = \int_0^{2\pi}\int_0^{2} 10\, e^{-\rho}\ \rho^2\, dz\, d\phi = 10 e^{-\rho}\ \rho^2 \times z \times (2\pi) = 10 e^{-2} \times 4 \times 4 \times (2\pi)$$

$$\oint_{S2} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = 43.3\pi$$

$$\oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = 0 - 48\pi + 43.3\pi = -4.7\pi$$

While the Left hand side is:

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{1}{\rho}\ \frac{\partial(\rho \mathbf{E}_\rho)}{\partial\rho} + \frac{1}{\rho}\ \frac{\partial \mathbf{E}_\phi}{\partial\phi} + \frac{\partial \mathbf{E}_z}{\partial z} = \frac{1}{\rho}\frac{\partial(\rho^2\ 10 e^{-\rho})}{\partial\rho} + \frac{1}{\rho}\ \frac{\partial(0)}{\partial\phi} + \frac{\partial(-3z)}{\partial z}$$

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{1}{\rho}(20\ \rho e^{-\rho} - \rho^2\ 10 e^{-\rho}) - 3 =$$

Therefore, the left hand side of the divergence theorem is given by:

$$\int_v (\vec{\nabla}\cdot\vec{\mathbf{E}})\,dv = \int_0^{2\pi}\int_0^{4}\int_0^{2}\left[(20\,e^{-\rho} - 10\rho\;e^{-\rho} - 3\right]\rho\,d\rho\,d\phi\,dz$$

$$first\quad term\;=\int_0^{2\pi}\int_0^{4}\int_0^{2} 20\,e^{-\rho}\;\rho\,d\rho\,d\phi\,dz = 160\pi\int_0^{2} e^{-\rho}\;\rho\,d\rho \qquad by\quad (u\,dv)$$

$$second\;\;term = -\int_0^{2\pi}\int_0^{4}\int_0^{2} 10\,e^{-\rho}\;\rho^2\;d\rho\,d\phi\,dz = -80\,\pi\int_0^{2} e^{-\rho}\;\rho^2\;d\rho \qquad by\quad (u\,dv)$$

$$third\quad term\quad = -\int_0^{2\pi}\int_0^{4}\int_0^{2} 3\,\rho\,d\rho\,d\phi\,dz = -\,24\,\pi\int_0^{2}\rho\,d\rho = -\,24\pi\times\left.\frac{\rho^2}{2}\right|_0^2 = -48\,\pi$$

Evaluating the value of the above integrals, and summing them, to get the actual Value of the left hand side, and it must be equal to the right side.

Evaluation of these integrals is left as an (Home Work)

Home Work

Q_1/ For the vector field $\vec{D} = 3r^2 \hat{a}_r$

, evaluate both sides of the divergence theorem for the region enclosed between the spherical shells defined by $r = 1\ m \quad and \quad r = 2\ m$

Q2/ Verify the divergence theorem for each of the following cases:

(a). For a vector $\vec{A} = x y^2 \hat{a}_x + y^2 \hat{a}_y + y^2 z \hat{a}_z$, and a surface defined by : $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$

(b). For a vector $\vec{A} = r^2 \hat{a}_r + r \sin\theta \cos\phi \hat{a}_\theta$, and surface defined by : $0 < r < 3\ ,\ 0 < \phi < \frac{\pi}{2}\ ,\ 0 < \theta < \frac{\pi}{2}$

(c). For a vector $\vec{A} = 2z\rho\ \hat{a}_\rho + 3z \sin\phi \hat{a}_\phi - 4\rho \cos\phi \hat{a}_z$, and surface defined by : $0 < \rho < 3, 0 < \phi < \frac{\pi}{4}, 0 < z < 5$

Q_3/ Evaluate both side of the divergence theorem for the field $\vec{F} = 2r^2 \hat{a}_r$ for the volume defined by: $0 \le r \le 3\ m\ ,\quad 0 \le \theta \le \pi \quad and \quad 0 \le \phi \le 2\pi$

Q_4/ Given $\vec{D} = \frac{30\ \rho^3}{4}\ \hat{a}_\rho$, evaluate both sides of the divergence theorem for the volume enclosed by: $0 \le \rho \le 2\quad ,\quad 0 \le z \le 10\ m \quad and \quad 0 \le \phi \le 2\pi$

Q5/ Determine the divergence of each of the following vector fields and then evaluate it at indicated point:

a. $\vec{E} = \rho \cos\phi \hat{a}_\rho + \rho \sin\phi \hat{a}_\phi + 3z \hat{a}_z \quad at \quad (2, 0^\circ, 3)$

b. $\vec{E} = 3x^2 \hat{a}_x + 2z \hat{a}_y + x^2 \hat{a}_z \quad at \quad (2, -2, 0)$