Chapter three: Electrostatic Field

3-1: The Electromagnetic Force:

The electromagnetic force consist of an electric force (F_e) and a magnetic force (F_m). The electric force is similar to gravitational force, but with a major difference:

(1). The source of gravitational force is mass, while the source of electric force is electric charge

(2). Both type of forces vary inversely as the square of the distance from the respective sources.

(3). Electric charge may have positive or negative polarity, whereas mass does not exhibit such a polarity.

Electric charge exhibits two important properties:

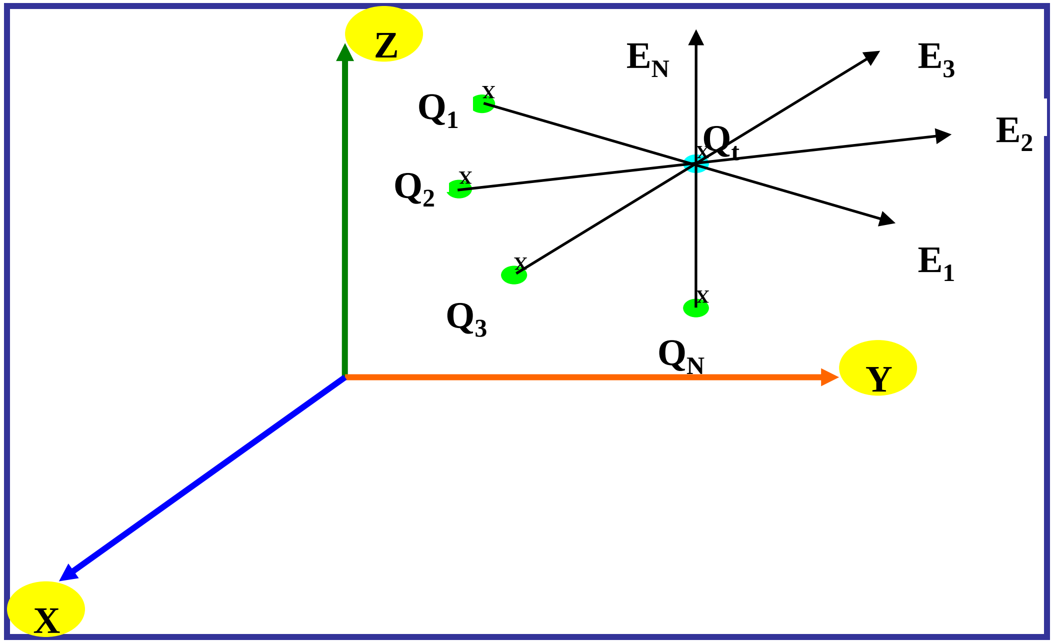
(1). The law of conservation of electric charge; which states that [the net electric charge can neither be created nor destroyed]. If a volume contains (n_p) protons and (n_e) electrons, then the total charge is:

$$q = n_p e - n_e e = (n_p - n_e) e$$

(2). The principle of linear superposition, which states that, [the total vector electric field at a point in space due to a system of point charges is equal to the vector sum of the electric fields at that point due to the individual charges]

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + \vec{\mathbf{E}}_4 + ------ + \vec{\mathbf{E}}_N$$

$$\vec{\mathbf{E}}_t = \frac{q_1}{4\pi\varepsilon_\circ R_1^2}\hat{a}_{R1} + \frac{q_2}{4\pi\varepsilon_\circ R_2^2}\hat{a}_{R2} + \frac{q_3}{4\pi\varepsilon_\circ R_3^2}\hat{a}_{R3} + \frac{q_4}{4\pi\varepsilon_\circ R_4^2}\hat{a}_{R4} + ----$$



3-2: Charge and Current Densities:

In electromagnetic, we encounter various forms of electric charge distributions and if the charges are in motion they constitute current distributions. Charge may be distributed over a volume of space, across a surface or along a line.

3-2-1: Charge Densities:

(١)Group of point charge: which is defined as the sum of the total point charges distributed non-uniformly over a given space :

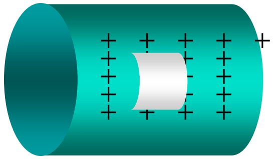
$$Q_t = Q_1 + Q_2 + Q_3 + - - - - Q_N = \sum_{k=1}^{N} Q_k$$

(٢)Line Charge Density: is defined as the total charges per unit length, which is distributed uniformly over a segment of line which need not be straight :

$$\rho_L = \frac{dq}{dl} \quad or \quad Q = \int \rho_L \; dl$$

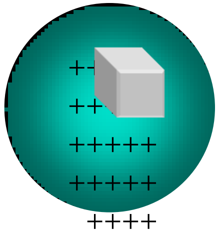
(3). Surface Charge Density: is defined as the total charge per unit area, which is distributed uniformly across the surface of a material:

$$\rho_S = \frac{dq}{ds} \quad or \quad Q = \int \rho_S \; ds$$



(4). Volume Charge Density: is defined as the total charge per unit volume, which is distributed uniformly over a volume of a given space:

$$\rho_V = \frac{dq}{dv} \quad or \quad Q = \int \rho_v \, dv$$



Example: Find the total charge contained in a cylindrical volume defined by : $\rho \le 2\ m \quad and \ \ 0 \le z \le 3\ m$, if $\rho_v = 10\ z\ \rho \ \ (mC/m^3)$?

Solution:

$$Q = \int_v \rho_v \, dv = \int_0^2 \int_0^{2\pi} \int_0^3 (10\, z\, \rho)\, \rho\, d\rho\, d\phi\, dz = 10 \int_0^3 z\, dz \int_0^{2\pi} d\phi \int_0^2 \rho^2\, d\rho$$

$$Q = 10 \left(\frac{z^2}{2} \right) \Big|_0^3 (\phi) \Big|_0^{2\pi} \left(\frac{\rho^3}{3} \right) \Big|_0^2 = \frac{20\pi}{6} (9-0) \times (8-0)$$

$$Q = 240\pi \ (C/m^3)$$

Home Work:

Q1 /Find the total charge contained in a cone defined by

$r \le 2\ m \ \ and \ \ 0 \le \theta \le 45^\circ$ **given that :** $\rho_v = 20\ r^2\ \cos^2\theta \quad (mC/m^3)$

Q2 /If the line charge density is given by $\rho_l = 12\ y^2\ (mC/m)$

Find the total charge distributed on the y-axis from $y = -5 \quad to \quad y = 5$?

Q3 /Find the total charge on a circular disk defined by: $\rho \le a \quad and \quad z = 0$ **if :**

a . $\rho_s = \rho_\circ \sin\phi \quad (C/m^2)$

b. $\rho_s = \rho_\circ \sin^2\phi \quad (C/m^2)$

c . $\rho_s = \rho_\circ e^{-r} \quad (C/m^2)$

d . $\rho_s = \rho_\circ e^{-r} \sin^2\phi \quad (C/m^2)$

3-2-2: Current Densities:

Consider a tube of charge with volume charge density (ρ_v), as shown in figure below. The charge are moving with a mean velocity ($\vec{\mathbf{u}}$) along the axis of the tube. Over a period (Δt) , the charges move a distance $(\Delta l = \vec{\mathbf{u}}\,\Delta t)$, then the amount of charge that crosses the cross-sectional (Δs) is:

$$\Delta q = \rho_v\,\Delta v = \rho_v\,\Delta l\,\Delta s = \rho_v\,\vec{\mathbf{u}}\,\Delta t\;\;\Delta s \;------(1)$$

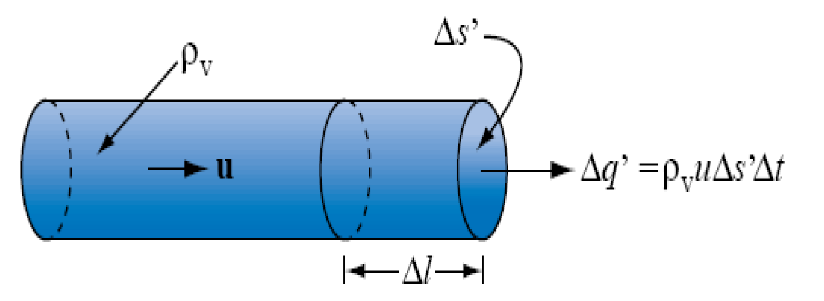
However, when the surface direction is not in parallel with($\vec{\mathbf{u}}$) then: $\overrightarrow{\Delta s} = \hat{a}_n\,\Delta s$

$$\Delta q = \rho_v\,\vec{\mathbf{u}}\cdot\hat{a}_n\,\Delta s\;\;\Delta t \;------(2)$$

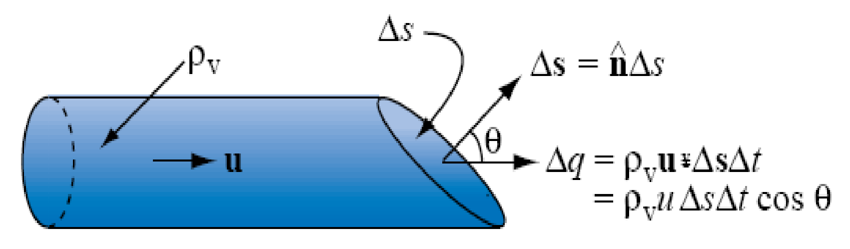
Then the corresponding current is :

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v\,\vec{\mathbf{u}}\cdot\hat{a}_n\,\Delta s = \vec{\mathbf{J}}\cdot\overrightarrow{\Delta s} \;-----(3)$$

Where, ($\vec{\mathbf{J}} = \rho_v\,\vec{\mathbf{u}}$), is defined as the current density in (A/m^2) , for arbitrary surface (S), the total current flowing through it is then given by: $I = \int \vec{\mathbf{J}}\cdot\overrightarrow{\mathbf{ds}}$



(a)



(b)

There are three types of current densities named convection, conduction and displacement currents).

(1). Convection Current Density ($\vec{\mathbf{J}}_v = \rho_v \vec{\mathbf{u}}$): is defined as a current which produced due to a movement of charged particle through a vacuum, air, or non-conductive media such as (beam of electron in a cathode ray tube or TV-screen).

(2). Conduction Current Density ($\vec{\mathbf{J}}_c = \sigma \vec{\mathbf{E}}$) : is defined as a current which produced due to a movement of electrons through conductive media in response to an applied electric field such as (the flow of current in Copper wires). It is given by the point form of Ohm's law as:

(٣). Displacement Current Density ($\vec{\mathbf{J}}_d = \dfrac{\partial \vec{\mathbf{D}}}{\partial t}$) : is defined as the time varying electric field phenomenon that allows current to flow between the plates of a capacitance.

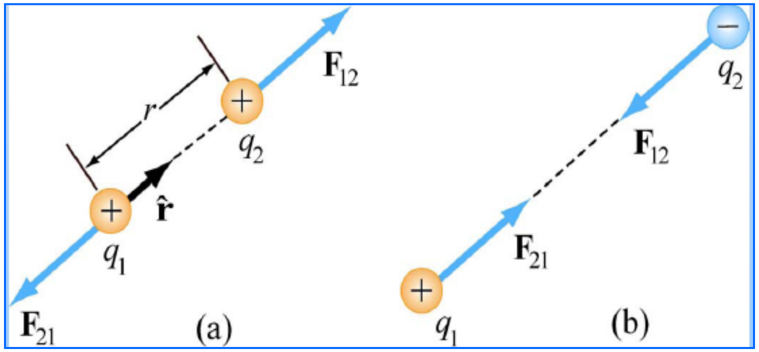
3-3: Force Between point Charges:

The electrostatic study begins with the first reported experiment of Coulomb's law in 1785 [French Colonel]. It deals with the force of point charge exerts on another point charge. The results of the experiments are expressed in Coulomb's law which states that: The force between two stationary point charges Q_1 and Q_2 is:

(1). along the line joining them,

(2). directly proportional to the product of charges Q1 Q2,

(3). inversely proportional to the square of the distance (R) between them.



Where mathematically is expressed as :

$$\vec{\mathbf{F}} = \frac{Q_1 Q_2}{4\pi\varepsilon_\circ R^2}\,\hat{a}_R \quad --------(1)$$

$$\vec{\mathbf{F}} = k\,\frac{Q_1 Q_2}{R^2}\,\hat{a}_R \quad , \quad where\ \ k = \frac{1}{4\pi\varepsilon_\circ} = 9\times10^{9}\ (N.m^2/C^2)$$

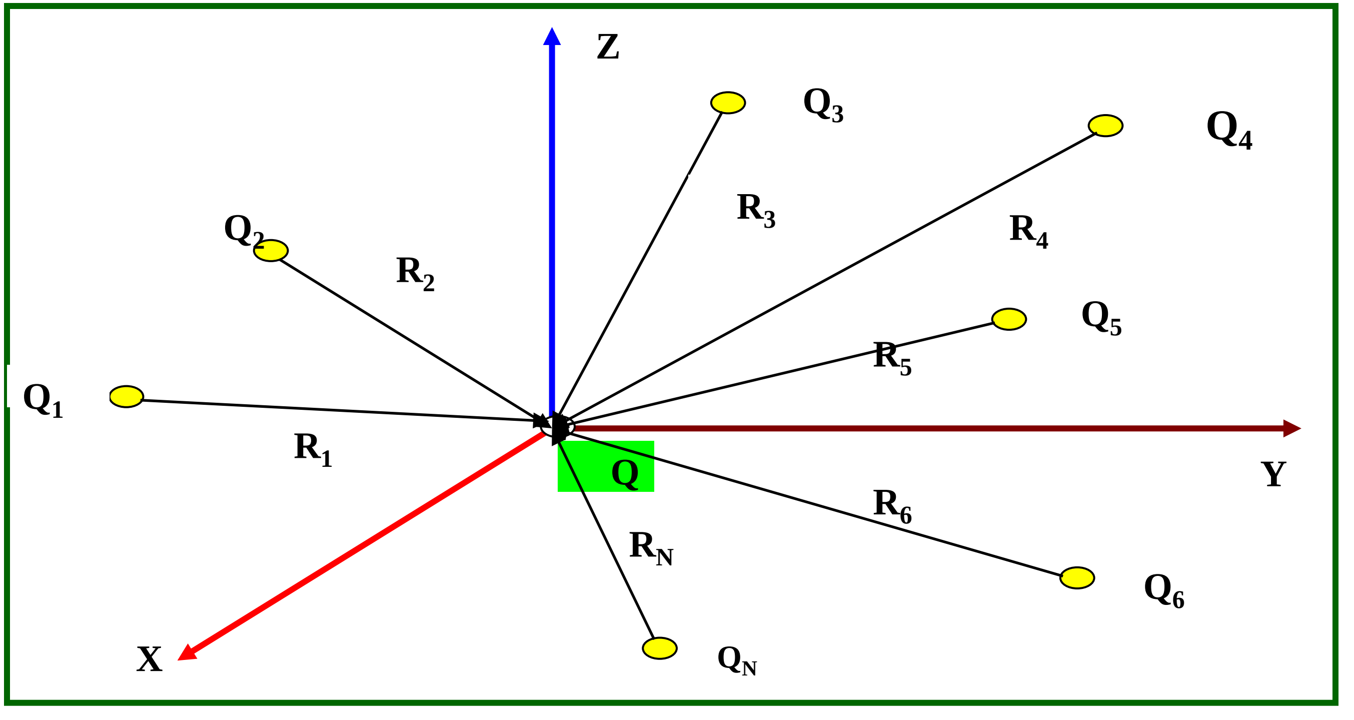
$$\varepsilon_\circ = \frac{10^{-9}}{36\pi}\ (F/m) = 8.85\times10^{-12}\ (F/m)$$

If point charges Q_1 and Q_2 are located at points having position vec $\vec{\mathbf{r}}_1$ *and* $\vec{\mathbf{r}}_2$, then the force ($\vec{\mathbf{F}}_{12}$) on Q_2 due to Q_1 is given by:

$$\vec{\mathbf{F}}_{12} = k\frac{Q_1 Q_2}{R_{12}^2}\hat{a}_{R_{12}} --------(2)$$

$$\vec{R}_{12} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 \qquad \left|\vec{R}_{12}\right| = R_{12}$$

$$\hat{a}_{R_{12}} = \frac{\vec{R}_{12}}{\left|\vec{R}_{12}\right|} \qquad \vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$



$$\vec{\mathbf{F}}_t = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + ------ + \vec{\mathbf{F}}_N = \frac{Q}{4\pi\varepsilon}\left(\frac{Q_1}{R_1^2}\hat{a}_{R1} + \frac{Q_2}{R_2^2}\hat{a}_{R2} + \frac{Q_3}{R_3^2}\hat{a}_{R3} + ---- + \frac{Q_N}{R_N^2}\hat{a}_{RN}\right)$$

Example(2): Two point charges, $Q_1 = 50\,\mu C \quad and \quad Q_2 = 10\,\mu C$, are located at $(-1,1,-3)m \; and \; (3,1,0)m$ respectively, find the force on (Q_1)?

Solution:

$$\vec{\mathbf{F}}_{21} = \frac{Q_1\, Q_2}{4\pi\varepsilon R_{21}^2}\, \hat{a}_{R21}$$

Where, $\vec{R}_{21} = (-1-3)\hat{a}_x + (1-1)\hat{a}_y + (-3-0)\hat{a}_z$ **and** $|R_{21}| = \sqrt{(-4)^2 + (0)^2 + (-3)^2} = 5\,m$

$$\hat{a}_{R21} = \frac{\vec{R}_{21}}{|R_{21}|} = \frac{-4\hat{a}_x - 3\hat{a}_z}{5}$$

$$\vec{\mathbf{F}}_{21} = 9\times 10^9 \,\frac{50\times 10\times 10^{-12}}{25}\,\left(\frac{-4\hat{a}_x - 3\hat{a}_z}{5}\right)$$

$$\vec{\mathbf{F}}_{21} = 0.18(-0.8\hat{a}_x - 0.6\hat{a}_z)\; N$$

Home Work:

Q$_1$/ Two point charges, $Q_1 = 250\ \mu C$ and $Q_2 = -300\ \mu C$ are located at $(5,0,0)m$ and $(0,0,-5)m$, respectively. Find the force on Q_2 ?

Q$_2$/ Determine the force on a point charge $Q = 50\ \mu C$ located at $(0,0,5)m$ due to a point charge of $Q = 500\pi\ \mu C$ located at the origin ?

Q$_3$/ Three point charges, each with $Q = 3\ nC$ are located at the corners of a triangle in the Z-plane with one corner at the origin , another at $(2,0,0)m$, and the third at $(0,2,0)m$. Find the force acting on the charge located at the origin ?

Q4 /Four point charges, each with $Q = 20\ \mu C$ are on the x and y axes at ± 4 Find the force exerted by these charges on a $100\ \mu C$ point charge located at $(0,0,3)m$

Q5 /Ten identical charges of(٥٠٠) μC ،each are spaced equally around a circle of radius(٢)m .find the force on a charge of (٢٠-) μC located on the axis (٢ ،) from the plane of the circle ؟

Q6/ Identical charges of Q(C) are located at the eight corner of a cube with a side $l\,(m)$. Show that the coulomb force on each charge has a magnitude of

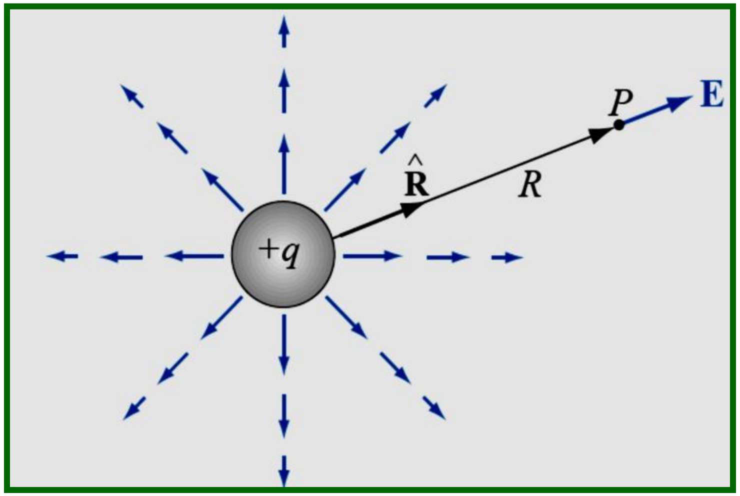
$$\left(\frac{3.29\, Q^2}{4\pi\varepsilon_\circ l^2}\ N \right)$$

3-4: Electric Field Intensity:

In the case of the gravitational field of a material body, we define the gravitational field intensity as the force per unit mass experienced by a small test mass placed in that field. In a similar manner the force per unit charge experienced by a small test charge placed in an electric field is known as electric field intensity $(\vec{\mathbf{E}})$

Alternatively, if in a region of space, a test charge (q) experiences a force $(\vec{\mathbf{F}})$, then the region is said to be characterized by an electric field $(\vec{\mathbf{E}})$ of intensity given by:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q} = \frac{1}{4\pi\varepsilon_\circ}\frac{q}{R^2}\,\hat{a}_R$$



$$d\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q} = \frac{1}{4\pi\varepsilon_\circ}\frac{dq}{R^2}\,\hat{a}_R \quad ----\text{point chareg}$$

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_\circ}\sum_{k=1}^{N}\frac{q_k}{R_k^2}\,\hat{a}_{Rk} \quad ----\text{Group of charge}$$

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_\circ}\int\frac{\rho_L\,dl}{R^2}\,\hat{a}_R \quad ----\text{Line charge dist.}$$

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_\circ}\int\frac{\rho_S\,ds}{R^2}\,\hat{a}_R \quad ----\text{surface charge dist.}$$

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_\circ}\int\frac{\rho_v\,dv}{R^2}\,\hat{a}_R \quad ----\text{Volume charge dist.}$$

3-4-1: Electric Field due to a Group of Charges:

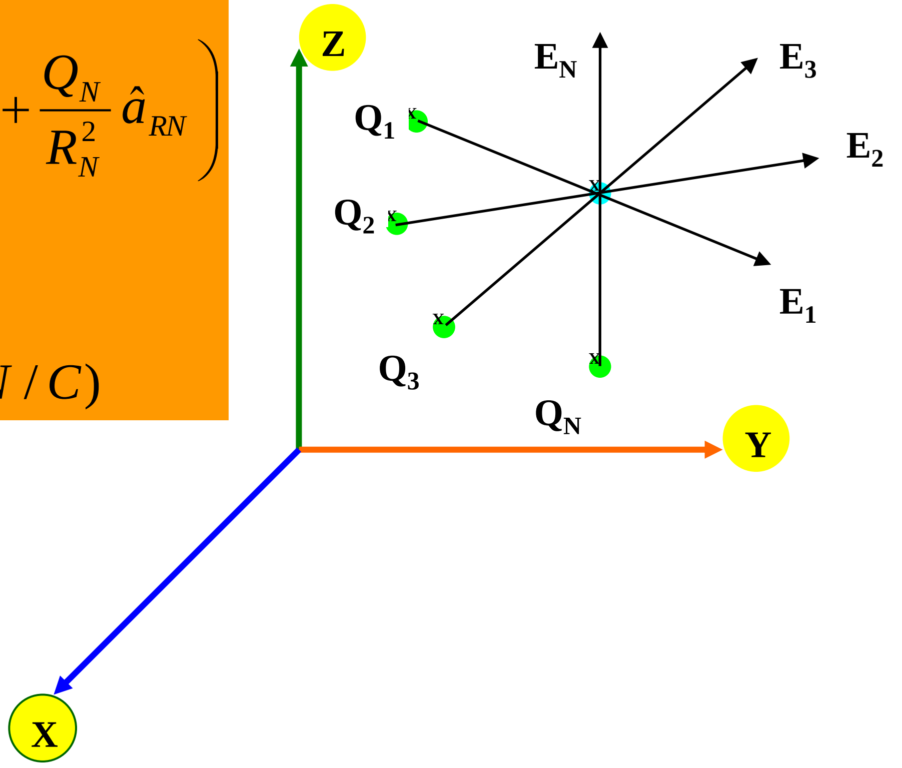
The electric field intensity of (N) point charges at a given point in space is equal to the vector sum of the electric field intensities $(\vec{\mathbf{E}})$ due to each charge acting alone, and is given by:

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + - - - - - \vec{\mathbf{E}}_N = \sum_{k=1}^{N} \vec{\mathbf{E}}_k$$

$$\vec{\mathbf{E}}_t = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1}{R_1^2}\hat{a}_{R1} + \frac{Q_2}{R_2^2}\hat{a}_{R2} + \frac{Q_3}{R_3^2}\hat{a}_{R3} + - - - - + \frac{Q_N}{R_N^2}\hat{a}_{RN}\right)$$

$$\vec{\mathbf{E}}_t = \sum_{k=1}^{N} \frac{Q_k}{4\pi\varepsilon R_k^2}\hat{a}_{Rk}$$

and it measure in unit of (V/m) *or* (N/C)



Example(3): A positive charge ($Q_1=10^{-9}C$) is located on the y-axis at ($y=2$) and a charge ($Q_2= - 10^{-9}C$) is located on the y-axis at ($y=-2$). Find the total force and electric field intensity on a small positive test charge (Q_t) located at point (10,0,0).

$$\vec{\mathbf{F}}_t = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2$$

$$\vec{\mathbf{F}}_1 = \frac{Q_1 Q_t}{4\pi\varepsilon R_1^2}\hat{a}_{R1} \quad and \quad \vec{\mathbf{F}}_2 = \frac{Q_2 Q_t}{4\pi\varepsilon R_2^2}\hat{a}_{R2}$$

$$\vec{\mathbf{R}}_1 = (10-0)\hat{a}_x + (0-2)\hat{a}_y + (0-0)\hat{a}_z = 10\hat{a}_x - 2\hat{a}_y$$

$$\vec{\mathbf{R}}_2 = (10-0)\hat{a}_x + (0+2)\hat{a}_y + (0-0)\hat{a}_z = 10\hat{a}_x + 2\hat{a}_y$$

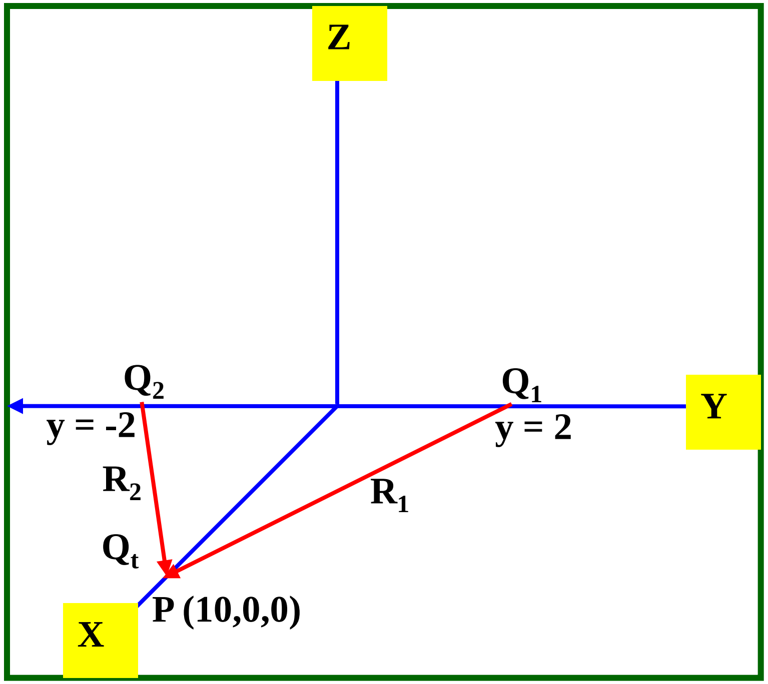
$$\hat{a}_{R1} = \frac{\vec{\mathbf{R}}_1}{\left|\vec{\mathbf{R}}_1\right|} = \frac{10\hat{a}_x - 2\hat{a}_y}{\sqrt{104}} \qquad \hat{a}_{R2} = \frac{\vec{\mathbf{R}}_2}{\left|\vec{\mathbf{R}}_2\right|} = \frac{10\hat{a}_x + 2\hat{a}_y}{\sqrt{104}}$$

$$\vec{\mathbf{F}}_t = \frac{Q_t}{4\pi\varepsilon_\circ}\left(\frac{Q_1}{R_1^3}(10\hat{a}_x - 2\hat{a}_y) + \frac{Q_2}{R_2^3}(10\hat{a}_x + 2\hat{a}_y)\right)$$

$$\vec{\mathbf{F}}_t = \frac{9\times10^9 \times 10^{-9}\, Q_t}{104\sqrt{104}}\left(-4\,\hat{a}_y\right)$$

$$\therefore \quad \vec{\mathbf{F}}_t = -\frac{36\,Q_t}{104\sqrt{104}}\hat{a}_y \qquad and \quad \vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{Q_t}$$

$$Then: \quad \vec{\mathbf{E}} = -\frac{36}{104\sqrt{104}}\hat{a}_y$$



Example(4): Find the total electric field intensity at the origin due to a (10^{-8}C) charge located at the point (0,4,4) and a ($-0.5 * 10^{-8}$)C charge located at point (4,0,4).

$$\vec{\mathbf{F}}_t = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2$$

$$\vec{\mathbf{F}}_1 = \frac{Q_1\,Q_t}{4\pi\,\varepsilon\,R_1^2}\,\hat{a}_{R1} \quad and \quad \vec{\mathbf{F}}_2 = \frac{Q_2\,Q_t}{4\pi\,\varepsilon\,R_2^2}\,\hat{a}_{R2}$$

$$\vec{\mathbf{R}}_1 = (0-0)\,\hat{a}_x + (0-4)\,\hat{a}_y + (0-4)\,\hat{a}_z = -4\,\hat{a}_y - 4\hat{a}_z$$

$$\vec{\mathbf{R}}_2 = (0-4)\,\hat{a}_x + (0-0)\,\hat{a}_y + (0-4)\,\hat{a}_z = -4\,\hat{a}_x - 4\hat{a}_z$$

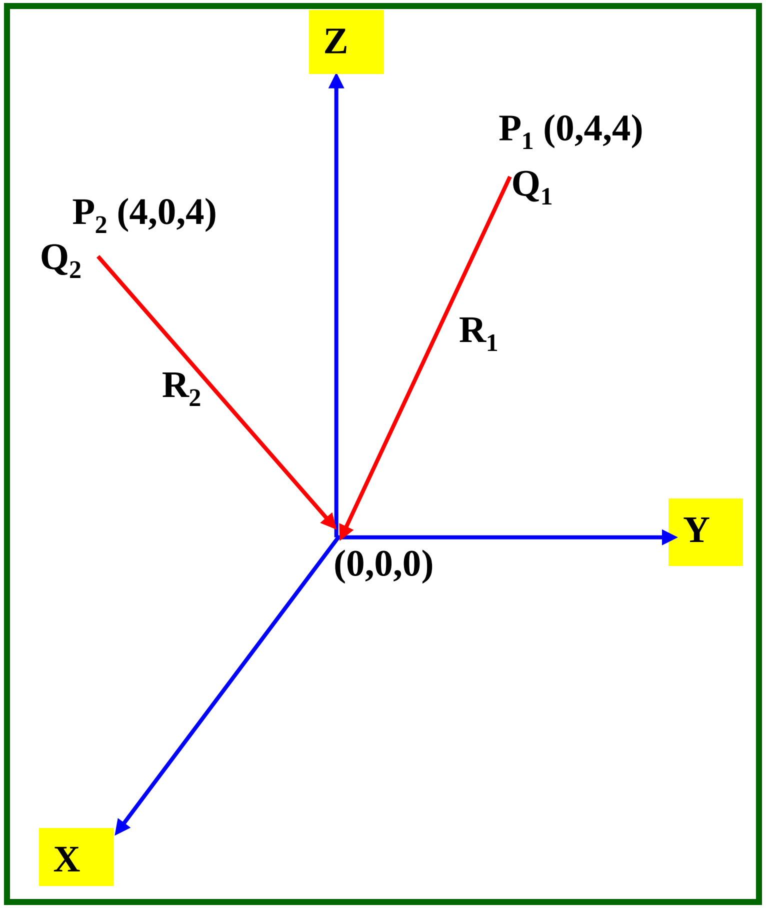
$$\hat{a}_{R1} = \frac{\vec{\mathbf{R}}_1}{\left|\vec{\mathbf{R}}_1\right|} = \frac{-4\,\hat{a}_y - 4\hat{a}_z}{\sqrt{32}} \qquad \hat{a}_{R2} = \frac{\vec{\mathbf{R}}_2}{\left|\vec{\mathbf{R}}_2\right|} = \frac{-4\,\hat{a}_x - 4\hat{a}_z}{\sqrt{32}}$$

$$\vec{\mathbf{F}}_t = \frac{Q_t}{4\pi\,\varepsilon_\circ}\left(\frac{Q_1}{R_1^3}(-4\,\hat{a}_y - 4\hat{a}_z) + \frac{Q_2}{R_2^3}(-4\,\hat{a}_x - 4\hat{a}_z)\right)$$

$$\vec{\mathbf{F}}_t = \frac{9\times10^9\times10^{-8}\;Q_t}{32\sqrt{32}}\left(-4\,\hat{a}_y - 4\hat{a}_z + 2\,\hat{a}_x + 2\hat{a}_z\right)$$

$$\therefore\;\; \vec{\mathbf{F}}_t = \frac{90\;Q_t}{32\,\sqrt{32}}(2\,\hat{a}_x - 4\,\hat{a}_y - 2\hat{a}_z)\,N \qquad and \quad \vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{Q_t}$$

$$Then:\;\; \vec{\mathbf{E}} = \frac{90\;Q_t}{32\,\sqrt{32}}(2\,\hat{a}_x - 4\,\hat{a}_y - 2\hat{a}_z)\;(N/C)$$

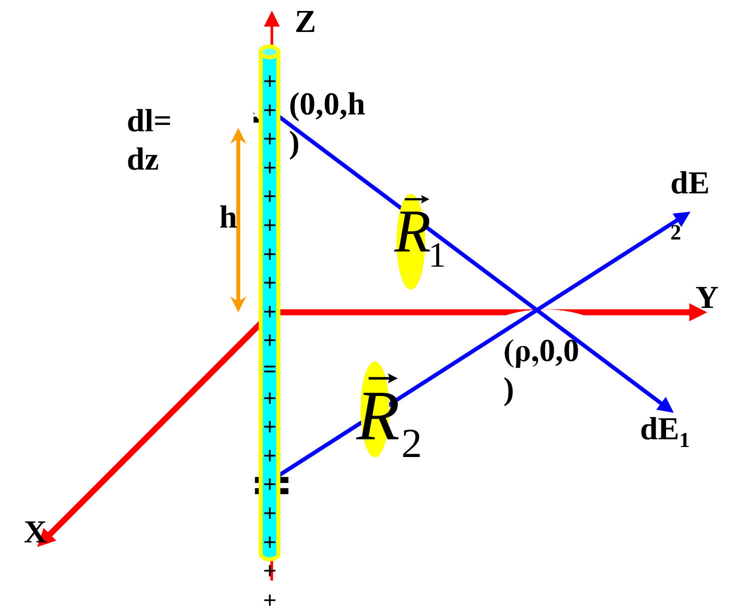


3-4-2: Electric Field Intensity of a Uniform Line Charge Distribution:

The uniform line charge distributions may be exist as straight line or curves. In the following the electric field intensity of a straight and circular (or ring) line charge distribution are determined with the help of vector calculus:

(1). Find the electric field intensity at the $z = 0 - plane$ due to a straight line charge distributions (ρ_L) located on the $z - axis$

Solution:



$$d\vec{\mathbf{E}}_t = d\vec{\mathbf{E}}_1 + d\vec{\mathbf{E}}_2 \quad --------(1) \qquad dq = \rho_L\, dl = \rho_L\, dz$$

$$d\vec{\mathbf{E}}_t = \frac{dq}{4\pi\varepsilon R_1^2}\hat{a}_{R1} + \frac{dq}{4\pi\varepsilon R_2^2}\hat{a}_{R2} \quad ------------(2)$$

$$\vec{\mathbf{R}}_1 = \rho\hat{a}_\rho - h\hat{a}_z \qquad |\vec{\mathbf{R}}_1| = \sqrt{\rho^2 + h^2} \qquad \hat{a}_{R1} = \frac{\vec{\mathbf{R}}_1}{|\vec{\mathbf{R}}_1|} = \frac{\rho\hat{a}_\rho - h\hat{a}_z}{\sqrt{\rho^2 + h^2}}$$

$$\vec{\mathbf{R}}_2 = \rho\hat{a}_\rho + h\hat{a}_z \qquad |\vec{\mathbf{R}}_2| = \sqrt{\rho^2 + h^2} \qquad \hat{a}_{R2} = \frac{\vec{\mathbf{R}}_2}{|\vec{\mathbf{R}}_2|} = \frac{\rho\hat{a}_\rho + h\hat{a}_z}{\sqrt{\rho^2 + h^2}}$$

$$d\vec{\mathbf{E}}_t = \frac{\rho_l}{4\pi\varepsilon_\circ}\left[\int_{-\infty}^{0}\frac{dz\;(\rho\hat{a}_\rho + h\hat{a}_z)}{(\rho^2 + z^2)^{3/2}} + \int_{0}^{\infty}\frac{dz\;(\rho\hat{a}_\rho - h\hat{a}_z)}{(\rho^2 + z^2)^{3/2}}\right] \quad -------(3)$$

$$let:\; z = \rho\tan\theta \qquad , \qquad dz = \rho\sec^2\theta\, d\theta$$

$$\vec{\mathbf{E}} = \frac{\rho_l}{4\pi\varepsilon_\circ}\int_{-\pi/2}^{\pi/2}\frac{\rho^2\sec^2\theta\; d\theta}{\rho^3\;\sec^3\theta}\hat{a}_\rho = \frac{\rho_l}{4\pi\varepsilon_\circ\rho}(\sin\theta)\Big|_{-\pi/2}^{\pi/2}\hat{a}_\rho$$

$$\vec{\mathbf{E}} = \frac{\rho_l}{2\pi\varepsilon_\circ\rho}\hat{a}_\rho \qquad or\;\; in\;\; general \qquad \vec{\mathbf{E}} = \frac{\rho_l}{2\pi\varepsilon_\circ r}\hat{a}_r$$

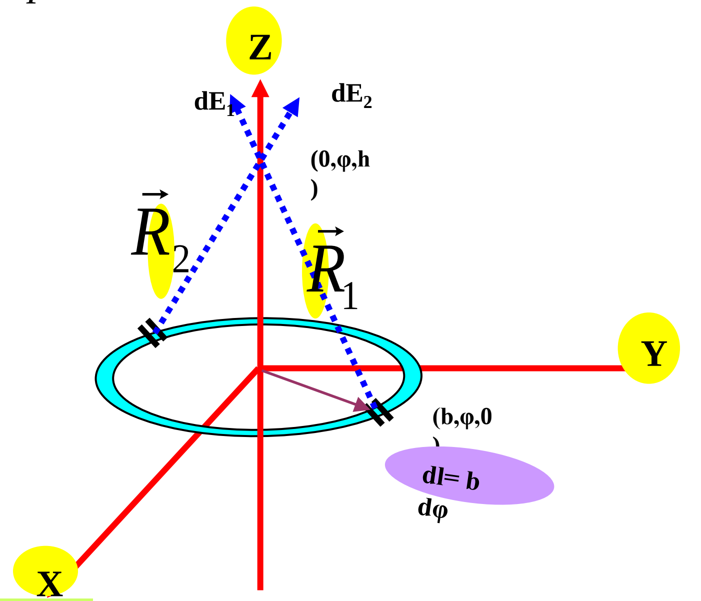
(2.) Find the electric field intensity on the $z-axis$ due to a line charge which is uniformly distributed over a ring with radius (b) located on the $z=0-plane$

$$d\vec{\mathbf{E}}_t = d\vec{\mathbf{E}}_1 + d\vec{\mathbf{E}}_2 \;--------(1) \qquad dq = \rho_L\, dl = \rho_L\; b\; d\phi$$

$$d\vec{\mathbf{E}}_t = \frac{dq}{4\pi\varepsilon_\circ R_1^2}\hat{a}_{R1} + \frac{dq}{4\pi\varepsilon_\circ R_2^2}\hat{a}_{R2} \;-------------(3)$$

$$\vec{\mathbf{R}}_1 = -b\,\hat{a}_\rho + h\,\hat{a}_z \qquad \left|\vec{\mathbf{R}}_1\right| = \sqrt{b^2+h^2} \qquad \hat{a}_{R1} = \frac{\vec{\mathbf{R}}_1}{\left|\vec{\mathbf{R}}_1\right|} = \frac{-b\,\hat{a}_\rho + h\,\hat{a}_z}{\sqrt{b^2+h^2}}$$

$$\vec{\mathbf{R}}_2 = b\,\hat{a}_\rho + h\,\hat{a}_z \qquad \left|\vec{\mathbf{R}}_2\right| = \sqrt{b^2+h^2} \qquad \hat{a}_{R2} = \frac{\vec{\mathbf{R}}_2}{\left|\vec{\mathbf{R}}_2\right|} = \frac{b\hat{a}_\rho + h\,\hat{a}_z}{\sqrt{b^2+h^2}}$$



$$d\vec{\mathbf{E}}_t = \frac{\rho_l}{4\pi\varepsilon_\circ}\left[\int_0^{\pi}\frac{b\,d\phi\;(-b\,\hat{a}_\rho + h\,\hat{a}_z)}{(b^2+z^2)^{3/2}} + \int_{\pi}^{2\pi}\frac{b\,d\phi\;(b\,\hat{a}_\rho + h\,\hat{a}_z)}{(b^2+z^2)^{3/2}}\right] -------(3)$$

$$\vec{\mathbf{E}} = \frac{\rho_l}{4\pi\varepsilon_\circ}\int_0^{2\pi}\frac{b\,h\;d\phi}{(b^2+h^2)^{3/2}}\hat{a}_z = \frac{\rho_l}{4\pi\varepsilon_\circ}\frac{b\,h}{(b^2+h^2)^{3/2}}(2\pi)\,\hat{a}_z = \frac{\rho_l}{2\,\varepsilon_\circ}\frac{b\,h}{(b^2+h^2)^{3/2}}\,\hat{a}_z$$

$$Q = 2\pi\rho_l\,b \quad , \quad then \quad \vec{\mathbf{E}} = \frac{Q\;h}{4\pi\varepsilon_\circ(b^2+h^2)^{3/2}}\hat{a}_z$$

$$and\;\; at\;\; the\;\; center\;\; of\;\; the\;\; ring\;\; (h=0) \quad \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_\circ b^3}\hat{a}_z$$

$$while \quad at \;\; (b=o) \quad \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_\circ h^2}\hat{a}_z \quad as \;\; of \;\; point\; charge$$

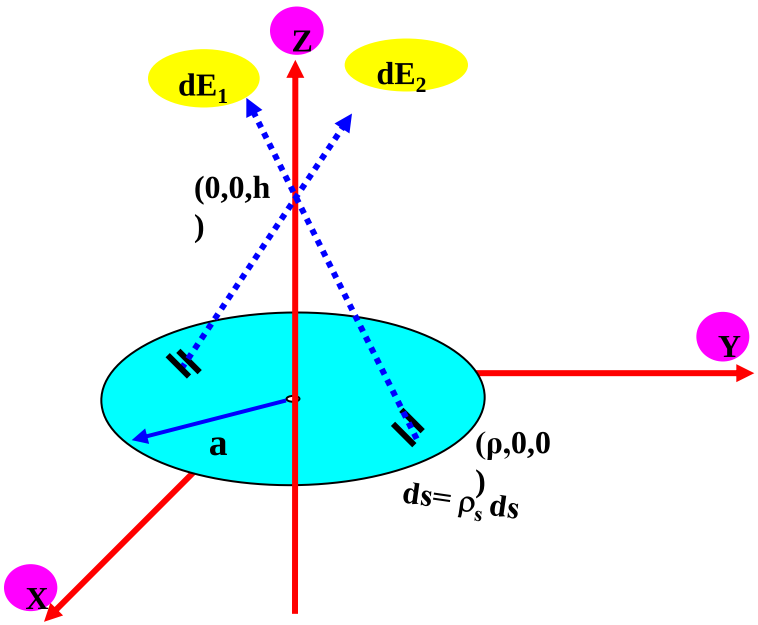
3-4-3: Electric Field Intensity of a Uniform surface Charge Distribution:

Find the electric field intensity at point $P(o,o,h)$ **in free space at a height (h) on the** $z-axis$ **due to a circular charge disk of radius (a) placed in the** $(xy-plane)\ or\ z=o\ plane$

with uniform charge density ρ_S **and then evaluate** $(\vec{\mathbf{E}})$ **for the infinite sheet case by letting** $a \to \infty$

$$d\vec{\mathbf{E}} = \frac{dq}{4\pi\varepsilon_\circ R^2}\hat{a}_R = \frac{\rho_s\, ds}{4\pi\varepsilon_\circ R^2}\hat{a}_R \qquad and \qquad ds = \rho\, d\rho\, d\phi$$

$$\vec{\mathbf{R}} = -\rho\,\hat{a}_\rho + h\,\hat{a}_z \qquad \left|\vec{\mathbf{R}}\right| = \sqrt{\rho^2+z^2} \qquad \hat{a}_R = \frac{-\rho\,\hat{a}_\rho + h\,\hat{a}_z}{\sqrt{\rho^2+z^2}}$$



Due to symmetry the component of electric field intensity along $\rho-axis)\ or\ (\hat{a}_\rho)-component$ **cancel each other, and then:**

$$\vec{\mathbf{E}} = \frac{\rho_s}{4\pi\varepsilon_\circ}\int_0^{2\pi}\int_0^{a}\frac{\rho\ h\ d\rho\, d\phi}{(\rho^2+h^2)^{3/2}}\ \hat{a}_z = \frac{\rho_s\ h}{2\varepsilon_\circ}\int_0^a \rho\,(\rho^2+h^2)^{-3/2}\, d\rho\,\hat{a}_z$$

$$\vec{\mathbf{E}} = \frac{\rho_s\ h}{2\varepsilon_\circ}\times\frac{1}{2}\times\frac{1}{-1/2\quad(\rho^2+h^2)^{1/2}}\Bigg|_o^a = \frac{\rho_s\ h}{2\varepsilon_\circ}\left[\frac{1}{h}-\frac{1}{\sqrt{a^2+h^2}}\right]\hat{a}_z$$

$$at\ (a\to\infty)\quad \vec{\mathbf{E}} = \frac{\rho_s}{2\varepsilon_\circ}\ \hat{a}_z$$

Therefore, for any infinite sheet charges we can write the electric field intensity as:

$$\vec{\mathbf{E}} = \frac{\rho_s}{2\varepsilon_\circ}\ \hat{a}_n$$

Example(5): In free space, there is a point charge (Q= 8 nC) at (-2,0,0), a line charge (ρ_l =10 nC/m) at (y= - 9 m) , (x= 0) and a sheet of charges with (ρ_s =10 nC/m2) located at (z= -2 m). Determine the electric field intensity at the origin due to these charge configurations.

Solution:

The electric field intensity due to the point charge is given by:

$$\vec{\mathbf{E}}_Q = \frac{Q}{4\pi\varepsilon_\circ R_Q^2}\hat{a}_{RQ} \qquad \vec{\mathbf{R}}_Q = (0+2)\hat{a}_x + (0-0)\hat{a}_y + (0-0)\hat{a}_z = 2\,\hat{a}_x \qquad \hat{a}_{RQ} = \frac{\vec{\mathbf{R}}_Q}{\left|\vec{\mathbf{R}}_Q\right|} = \frac{2\,\hat{a}_x}{2} = \hat{a}_x$$

$$\vec{\mathbf{E}}_Q = 9\times10^9 \times 8\times10^{-9}\left(\frac{\hat{a}_x}{4}\right) \Rightarrow\Rightarrow \vec{\mathbf{E}}_Q = 18\,\hat{a}_x(N/C) ------(1)$$

The electric field intensity due to the uniform line charge is given by:

$$\vec{\mathbf{E}}_l = \frac{\rho_l}{2\pi\varepsilon_\circ R_l}\hat{a}_{Rl} \qquad \vec{\mathbf{R}}_l = (0-0)\hat{a}_x + (0+9)\hat{a}_y + (0-0)\hat{a}_z = 9\,\hat{a}_y \qquad \hat{a}_{RQ} = \frac{\vec{\mathbf{R}}_l}{\left|\vec{\mathbf{R}}_l\right|} = \frac{9\,\hat{a}_y}{9} = \hat{a}_y$$

$$\vec{\mathbf{E}}_l = 2\times9\times10^9 \times 10\times10^{-9}\left(\frac{\hat{a}_y}{9}\right) \Rightarrow\Rightarrow \vec{\mathbf{E}}_l = 20\,\hat{a}_y(N/C) -----(2)$$

The electric field intensity due to the uniform surface charge is given by:

$$\vec{\mathbf{E}}_s = \frac{\rho_s}{2\varepsilon_\circ}\hat{a}_n \qquad \sin ce \quad \hat{a}_n = \hat{a}_z \Rightarrow\Rightarrow therefore: \qquad \vec{\mathbf{E}}_s = \frac{\rho_s}{2\varepsilon_\circ}\hat{a}_z$$

$$\vec{\mathbf{E}}_s = \frac{10\times10^{-9}}{2\pi\dfrac{10^{-9}}{36\pi}}\hat{a}_z \Rightarrow\Rightarrow \vec{\mathbf{E}}_s = 180\,\hat{a}_z\ (N/C) --------(3)$$

Therefore, the total electric field intensity at the origin is the sum of the equations (1), (2) and (3) as given :

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_Q + \vec{\mathbf{E}}_l + \vec{\mathbf{E}}_s = \left[18\,\hat{a}_x + 20\,\hat{a}_z + 180\,\hat{a}_z\,\right] \quad (N/C)$$

Example(6): Two uniform charge distributions are as follows: a sheet of uniform charge density (ρ_s = -50 nC/m2) at (y= 2 m) and a uniform line charge density of (ρ_l =0.2 micro Coulomb/m) at (z= 2 m) (y= -1 m). At what point in the region will the electric field intensity be zero?

Solution:

The total electric field intensity due to these charge configurations is the sum of the electric field intensity due to line charge and surface charges which mathematically represented as:

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_l + \vec{\mathbf{E}}_s --------(1)$$

when the total electric field intensity is equal to zero means that: $0 = \vec{\mathbf{E}}_l + \vec{\mathbf{E}}_s \Rightarrow\Rightarrow -\vec{\mathbf{E}}_s = \vec{\mathbf{E}}_l --------(2)$

Hence, the electric field intensity produced by this line charge at a given point $P(x_p, y_p, z_p)$ is calculated as:

$$\vec{\mathbf{E}}_l = \frac{\rho_l}{2\pi\varepsilon_\circ R_l}\hat{a}_{Rl} \qquad \vec{\mathbf{R}}_l = (x_p - x)\hat{a}_x + (y_p + 1)\hat{a}_y + (z_p - 2)\hat{a}_z$$

$$\left|\vec{\mathbf{R}}_l\right| = R_l = \sqrt{(x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2} \qquad \hat{a}_{Rl} = \frac{\vec{\mathbf{R}}_l}{\left|\vec{\mathbf{R}}_l\right|} = \frac{x_p\hat{a}_x + (y_p + 1)\hat{a}_y + (z_p - 2)\hat{a}_z}{\sqrt{(x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2}}$$

$$\vec{\mathbf{E}}_l = \frac{\rho_l}{2\pi\varepsilon_\circ\sqrt{(x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2}}\,\frac{(x_p - x)\hat{a}_x + (y_p + 1)\hat{a}_y + (z_p - 2)\hat{a}_z}{\sqrt{(x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2}}$$

$$\vec{\mathbf{E}}_l = \frac{\rho_l}{2\pi\varepsilon_\circ}\,\frac{(x_p - x)\hat{a}_x + (y_p + 1)\hat{a}_y + (z_p - 2)\hat{a}_z}{\left((x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2\right)} = \frac{0.2\times 10^{-6}}{2\pi\frac{10^{-9}}{36\pi}}\,\frac{(x_p - x)\hat{a}_x + (y_p + 1)\hat{a}_y + (z_p - 2)\hat{a}_z}{\left((x_p - x)^2 + (y_p + 1)^2 + (z_p - 2)^2\right)}$$

$$\vec{\mathbf{E}}_l = 3.6\times10^3 \frac{(x_p - x)\hat{a}_x + (y_p+1)\hat{a}_y + (z_p-2)\hat{a}_z}{\left((x_p-x)^2 + (y_p+1)^2 + (z_p-2)^2\right)} \quad ----------(3)$$

Hence, the electric field intensity produced by this sheet charge at a given point $P(x_p, y_p, z_p)$ is calculated as:

$$\vec{\mathbf{E}}_s = \frac{\rho_s}{2\varepsilon_\circ}\hat{a}_n \quad \text{since } \hat{a}_n = -\hat{a}_y \qquad \text{hence}: \vec{\mathbf{E}}_s = \frac{\rho_s}{2\varepsilon_\circ}(-\hat{a}_y) = \frac{-50\times10^{-9}}{2\frac{10^{-9}}{36\pi}}(-\hat{a}_y)$$

$$\vec{\mathbf{E}}_s = 900\pi\,\hat{a}_y \quad ----------(4)$$

Substituting eqs.(3) and (4) into eq.(2) we get:

$$\vec{\mathbf{E}}_l = -\vec{\mathbf{E}}_s \Rightarrow\Rightarrow -900\pi\,\hat{a}_y = 3.6\times10^3 \frac{(x_p - x)\hat{a}_x + (y_p+1)\hat{a}_y + (z_p-2)\hat{a}_z}{\left((x_p-x)^2 + (y_p+1)^2 + (z_p-2)^2\right)} \quad ------(5)$$

Equating the component of (x) (y) and (z) from both side of the equation (5) we obtain:

$$0\;\hat{a}_x = 3.6\times10^3 \frac{(x_p - x)\hat{a}_x}{\left((x_p-x)^2 + (y_p+1)^2 + (z_p-2)^2\right)} \Rightarrow\Rightarrow x_p - x = 0 \Rightarrow x_p = x$$

$$0\,\hat{a}_z = 3.6\times10^3 \frac{(z_p - 2)\hat{a}_z}{\left((x_p-x)^2 + (y_p+1)^2 + (z_p-2)^2\right)} \Rightarrow\Rightarrow z_p - 2 = 0 \Rightarrow z_p = 2$$

$$-900\pi\,\hat{a}_y = 3.6\times10^3 \frac{(y_p+1)\hat{a}_y}{\left((x-x)^2 + (y_p+1)^2 + (2-2)^2\right)} \Rightarrow \frac{-900}{3.6\times10^3} = \frac{(y_p+1)}{(y_p+1)^2}$$

$$(y_p+1) = \frac{3.6\times10^3}{-900\pi} = -1.273$$

$$y_p = -2.273$$

Therefore, the coordinate of the point at which the electric field intensity is zero due to these charge density configurations is :

$$(x_p, y_p, z_p) = (x, -2.273,\ 2)m$$

Home Work

Q$_1$/ An infinitely long line charge $\rho_L = 21\pi \frac{nC}{m}$ lies along the z-axis. An infinite area sheet charge $\rho_S = 3\frac{nC}{m^2}$ lies in the xz-plane (y=o). Find a point on the y-axis where the electric field intensity is zero ?

Q$_2$/ Plane $z = 10m$, carries charge $\rho_S = 20\frac{nC}{m^2}$. The electric field intensity at the origin is :

a- $-10\hat{a}_z \frac{v}{m}$ **b-** $-72\pi\hat{a}_z \frac{v}{m}$ **c-** $-18\pi\hat{a}_z \frac{v}{m}$ **c-** $360\pi\hat{a}_z \frac{v}{m}$

Q$_3$/ A point charge (100)pC is located at (4,1,-3), while the x-axis carries line charge $\rho_L = 2\frac{nC}{m}$. If the plane $z = 3m$ also carries surface charge of $\rho_S = 5\frac{nC}{m^2}$, then find the electric field intensity $\vec{E}$ due to these charges at point (1,1,1) ?

Q$_4$/ Line $x = 3$, $z = -1$ carries line charge $\rho_L = 20\frac{nC}{m}$, while plane $x = -2$ carries surface charge of $\rho_S = 4\frac{nC}{m^2}$. Find the force on a point charge $Q = -5mC$ located at the origin due these two charge distributions ?

Q$_5$/ Charge $Q_1 = 4\ \mu C$ is located at $(1,1,0)\,cm$ and charge Q_2 is located at $(0,0,4)\,cm$. What should Q_2 be so that $\vec{E}$ at $(0,2,0)\,cm$ has no y-component?

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Q$_6$/ A uniform charge distribution, infinite in extent, lies along the z-axis with $\rho_L = 20\frac{nC}{m}$. Find the electric field $\vec{E}$ at (6,8,3)m, expressing it in both Cartesian and cylindrical coordinates.

Q$_7$/ Two identical uniform line charges of $\rho_L = 4\frac{nC}{m}$, are parallel to the z-axis at $x = 0\ ,\ y = \mp 4\ m$. Determine the electric field $\vec{E}$ at (±4,0,z). Ans.($\pm 18\,\hat{a}_x\ V/m$).

Q$_8$/ Two identical uniform line charges of $\rho_L = 5\frac{nC}{m}$ are parallel to the x-axis, one at $z = 0\ ,\ y = -2\,m$ and the other at $z = 0\ ,\ y = 4\,m$. Find the $\vec{E}$ at (4,1,3) ? Ans. ($30\,\hat{a}_z\ V/m$).

Q$_9$/ Determine $\vec{E}$ at the origin due to a uniform line charge distribution with $\rho_L = 3.3\frac{nC}{m}$ located at $x = 3m\ ,\ y = 4\ m$?Ans.($-7.13\hat{a}_x - 9.5\hat{a}_y\ \ V/m$).

Q$_{10}$/ Two meters from the z-axis, the value of electric field intensity due to a uniform line charge along the z-axis is known to be $(1.8\times 10^4\ V/m)$. Find the value of this line charge density ρ_L?(2)μC.