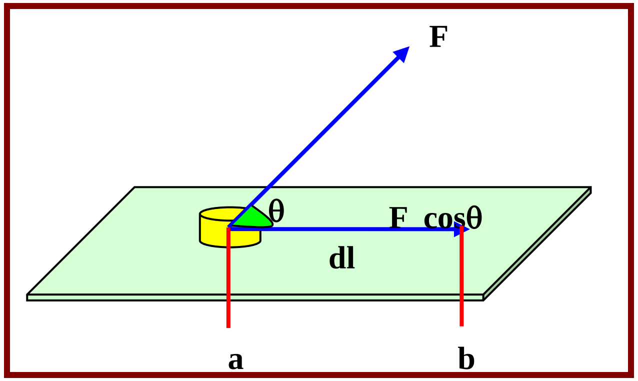
3-8: Electric Potential:

Form our discussion in the preceding sections; the electric field intensity ($\vec{E}$) due to a charge distribution can be obtained from Coulomb's Law in general or from Gauss's law when the charge distribution is symmetric.

Another way of obtaining ($\vec{E}$) is from the electric scalar potential (V) to be defined in this section. In a sense, this way of finding ($\vec{E}$) is easier because it is easier to handle scalar function than vectors.

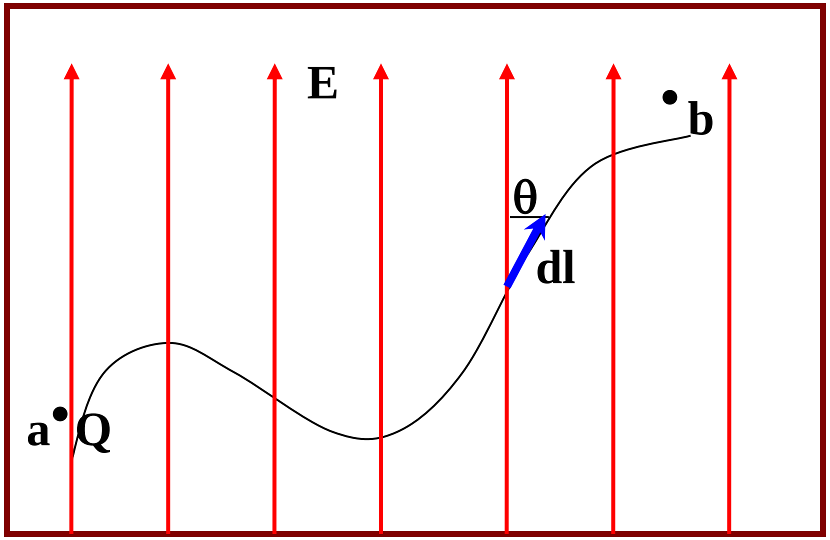
In this section we develop the concept of electric potential and show its relationship to electric field intensity. When force is applied to move an object, work is the product of the force and the distance the object travels in the direction of the force. Mathematically, in moving the object from point (a) to point (b), the work can be expressed as:

$$W = \int_a^b \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dl}} \quad ----------(1)$$



We know from Coulomb's law that the force exerted on a charge (Q) by an electric field ($\vec{E}$) is (F=Q E). Thus, the work done by the field in moving the charge from point (a) to point (b) is then:

$$W_{E-field} = \int_a^b \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dl}} = Q \int_a^b \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} \quad ---------(2)$$



If an external force moves the charge against the field, the work done is negative of $W_{E-field}$ or:

$$W_{ext.} = -W_{E-field} = -Q \int_a^b \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} \quad ---(3)$$

Dividing (W) by (Q) in eq.(3), gives the potential energy per unit charge. This quantity denoted by (V_{ab}), is known as the potential difference between points (a) and (b). Thus:

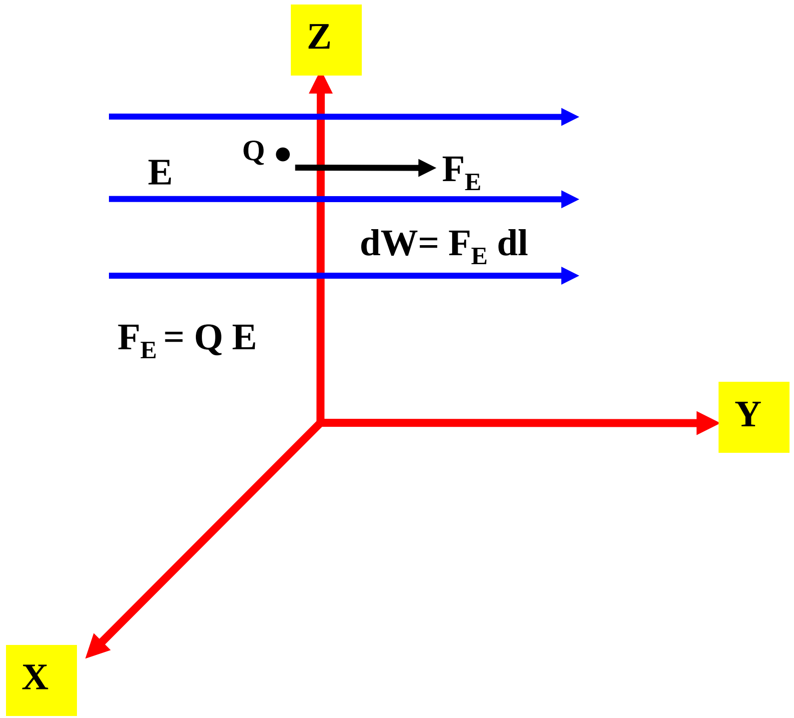
$$V_{ab} = \frac{W}{Q} = -\int_a^b \vec{\mathbf{E}} \cdot \vec{\mathbf{dl}} \quad ----------(4)$$

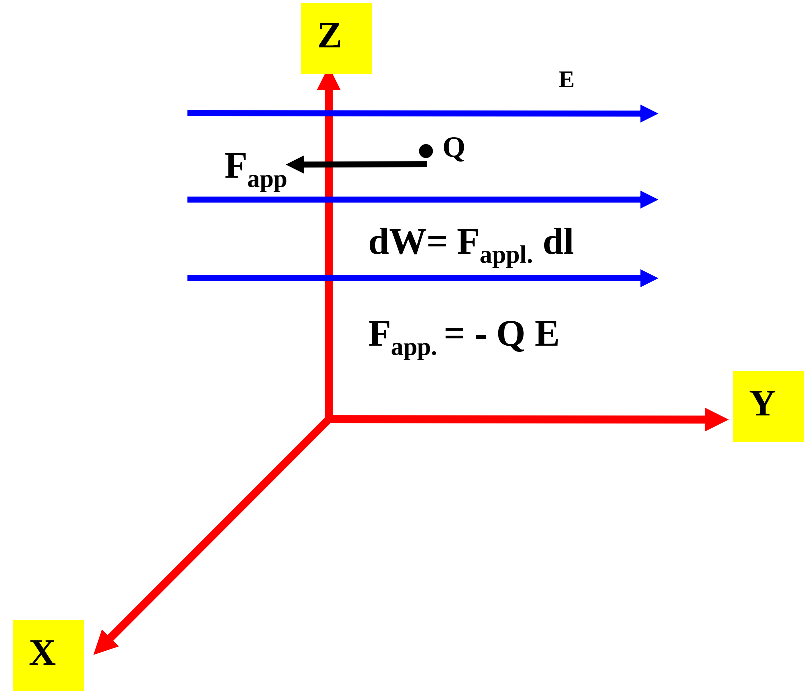
, and have the following properties:

(1). V_{ab} is independent of the path taken,

(2). V_{ab} is measured in unit of (J/C) or volts (V)

(3). In determining V_{ab} , (a) is the initial point while (b) is the final point.





$$\vec{\mathbf{dl}} = \begin{cases} dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z & --------- \ cartesian \\ d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z & --------- \ cylindrical \\ dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi & ------ spherical \end{cases}$$

$$V_{ab} = \begin{cases} - \ work \ \ performed \ by \ the \ \ field \\ 0 \ \ closed \ \ path \\ + \ work \ \ performed \ by \ an \ external \ \ force \end{cases}$$

$V_{ab} = V_a - V_b$

V_a is intial point

V_b is final point

Electric potential difference: **is defined as the work done by an external force to move a charge from point (a) to point (b) in an electric field divided by the amount of charge moved, and it related to absolute potential or electrostatic potential as :**

$$V_{ab} = V_b - V_a \ ------------ (5)$$

Absolute potential: **is defined as a potential at which the reference point is taken at infinity that is: ($V_\infty = zero$)**

It is interesting to see that, if a closed path is chosen the integral will return zero potential differences:

$$V_{abcda} = -\left[\int_a^b \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_b^c \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_d^a \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}}\right] = 0 ------(6)$$

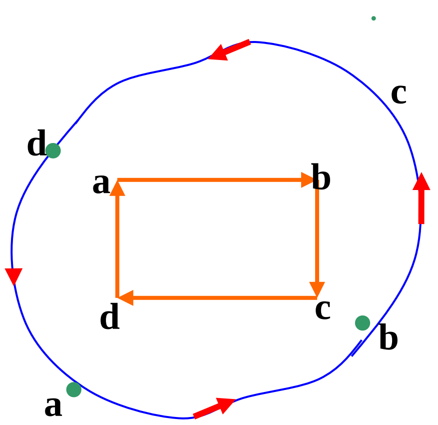
$$\textit{Then for closed path} \oint_C \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0 -------------(7)$$

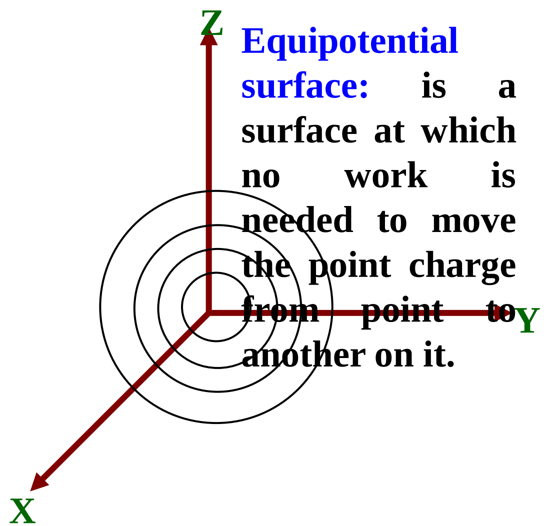
Equation (7) is also called Kirchhoff's voltage law. Physically this equation means that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes's theorem we get:

$$\oint_c \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = \oint_s \vec{\nabla}\times\vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{ds}} = 0$$

and hence:

$$\vec{\nabla}\times\vec{\mathbf{E}} = 0 ------------(8)$$





Equipotential surface: is a surface at which no work is needed to move the point charge from point to another on it.

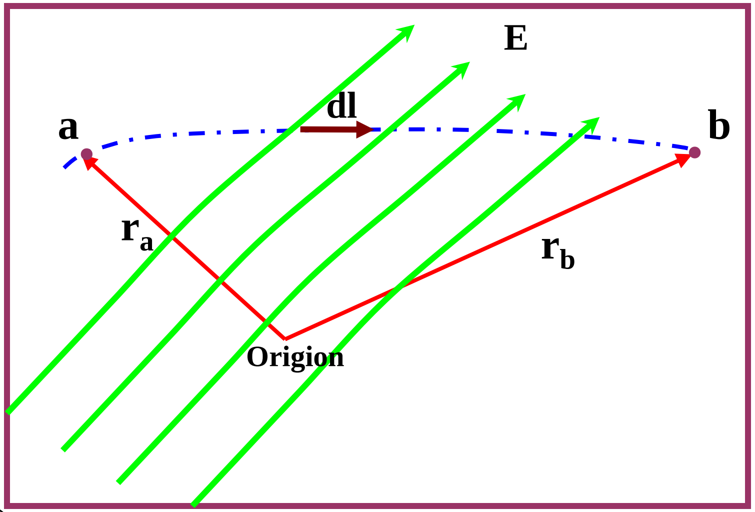
Eq.(8) **is called second Maxwell's equation in point form and in static electric field.** **This equation physically means that the electrostatic field is a conservative or ir-rotational field.**

$$\vec{\nabla} \times \vec{\mathbf{E}} = 0$$

$$V_{ab} = -\int_a^b \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} \quad ----------(4)$$

if the $\vec{\mathbf{E}} - field$ **is due to a point charge located at the origin** $(0,0,0)$

, then generally the electric field is given by:



$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon r^2}\,\hat{a}_r \quad --------(9)$$

$$V_{ab} = -\int_{r_a}^{r_b} \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\int_{r_a}^{r_b} \frac{Q}{4\pi\varepsilon r^2}\,\hat{a}_r \cdot dr\,\hat{a}_r = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) = V_b - V_a \quad ----------(10)$$

Thus the potential difference (V_{ab}) may be regarded as the potential at (b) with reference to (a). For point charge, it is customary to choose infinity as a reference that is we assume the potential at infinity is zero. Thus if ($V_a = 0 \ at \ \ a \to \infty$)

then the potential at any point $(r_b \to r)$ due to a point charge (Q) located at the origin is given by:

$$V = \frac{Q}{4\pi\varepsilon r} \qquad for \ \ point \ charge$$

$$V = \sum_{k=1}^{N} \frac{Q_k}{4\pi\varepsilon r_k} \qquad for \ \ a \ group \ of \ charge$$

$$V = \frac{1}{4\pi\varepsilon} \int \frac{\rho_l \ dl}{r} \qquad for \ line \ charge \ distribution$$

$$V = \frac{1}{4\pi\varepsilon} \int \frac{\rho_s \ ds}{r} \qquad for \ surface \ charge \ distribution$$

$$V = \frac{1}{4\pi\varepsilon} \int \frac{\rho_v \ dv}{r} \qquad for \ volume \ charge \ distribution$$

Example(1): A total charge of (Q) is uniformly distributed around a circular ring of radius (ρ). Find the potential at a point on the axis (h) m from the plane of the ring. Compare with the result where all charge is at the origin in the form of a point charge.

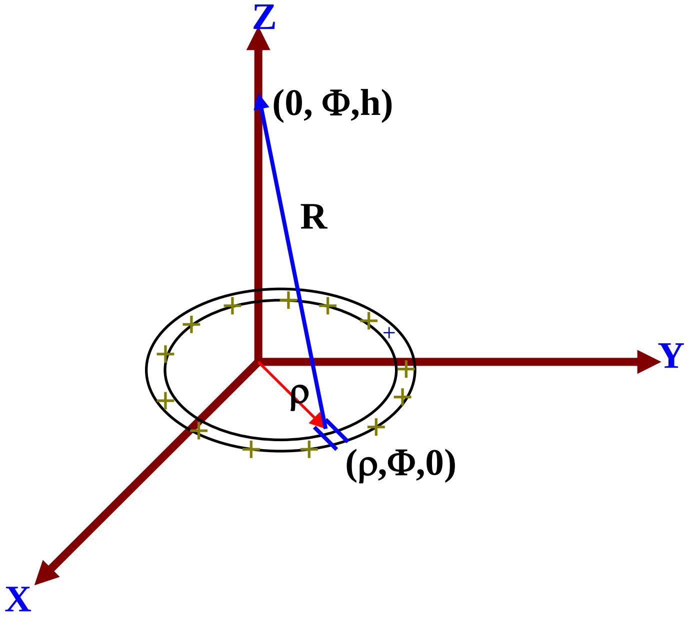
Solution:

$$\vec{\mathbf{R}} = -\rho\,\hat{a}_\rho + h\,\hat{a}_z \qquad \left|\vec{\mathbf{R}}\right| = \sqrt{\rho^2 + h^2} \qquad Q = \rho_l\, dl \qquad dl = \rho\, d\phi$$

$$V = \int \frac{\rho_l\, dl}{4\pi\varepsilon R} = \int_0^{2\pi} \frac{\rho_l\, \rho\, d\phi}{4\pi\varepsilon\sqrt{\rho^2 + h^2}} = \frac{\rho_l\, \rho}{2\varepsilon\sqrt{\rho^2 + h^2}}$$

$$Q = \rho_l\, dl = \rho_l\,(2\pi\rho) \Rightarrow \rho_l\, \rho = \frac{Q}{2\pi}$$

$$\therefore V = \frac{Q}{4\pi\varepsilon\sqrt{\rho^2 + h^2}}$$



If the charge is concentrated at the origin , then:

$$\rho = 0 \quad , \quad and \quad V = \frac{Q}{4\pi\varepsilon h}$$

Example(2): A total charge of (Q) is uniformly distributed over a circular disk of radius (ρ). Find the potential at a point on the axis (h)m from the plane of the disk.

Solution:

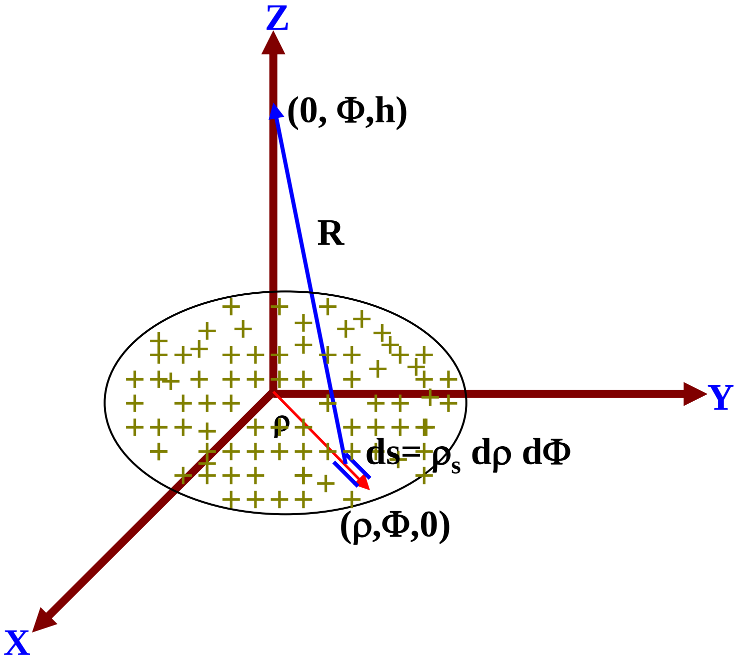
$$\vec{\mathbf{R}} = -\rho\,\hat{a}_\rho + h\,\hat{a}_z \qquad |\vec{\mathbf{R}}| = \sqrt{\rho^2 + h^2} \qquad Q = \rho_s\, ds \qquad ds = \rho\; d\rho\, d\phi$$

$$V = \int \frac{\rho_s\, ds}{4\pi\varepsilon R} = \int_0^{2\pi}\int_o^a \frac{\rho_s\;\rho\;\, d\rho\; d\phi}{4\pi\varepsilon\sqrt{\rho^2+h^2}} \qquad let \quad \rho = h\tan\theta \quad \Rightarrow \quad d\rho = h\sec^2\theta\, d\theta$$

$$V = \int_0^{2\pi}\int_o^a \frac{\rho_s\; h^2\tan\theta\,\sec^2\theta\;\; d\theta\; d\phi}{4\pi\varepsilon\; h\sec\theta} = \frac{\rho_s\; h}{4\pi\varepsilon}(2\pi)\int_0^a \tan\theta\,\sec\theta\, d\theta = \frac{\rho_s\; h}{2\varepsilon}(\sec\theta)$$

$$V = \frac{\rho_s\; h}{2\varepsilon}\;\frac{\sqrt{\rho^2+h^2}}{h}\;\Big|_0^a \qquad \Rightarrow \quad V = \frac{\rho_s}{2\varepsilon}\sqrt{\rho^2+h^2}\;\Big|_0^a$$

$$V = \frac{\rho_s}{2\varepsilon}\left(\sqrt{a^2+h^2} - h\right)$$



But in the ($z = 0\;\; plane$) the potential difference is reduces to:

$$V = \frac{\rho_s}{2\varepsilon}\,a$$

Example(3): Find the work done in moving $10\ nC$ charge from point $(2,\pi,\pi/2)$ to the origin in a field: $\vec{E}=\frac{50r}{r^2+1}\hat{a}_r\ \ (V/m)$?

Solution:

$$W=-Q\int \vec{\mathbf{E}}\cdot\vec{\mathbf{dl}}=-Q\int\frac{50r}{r^2+1}\hat{a}_r\cdot(dr\,\hat{a}_r+r\,d\theta\,\hat{a}_\theta+r\sin\theta\,d\phi\,\hat{a}_\phi)$$

$$W=-Q\int_2^0\frac{50r}{r^2+1}dr=-Q\times25\times\left(\ln(r^2+1)\right)\Big|_2^0=-25\times10\times10^{-9}\,(0-1.61)$$

$$W=402.36\ nJ$$

Example(4): Find the difference in the amounts of work required to bring a point charge $Q=2nC$ from infinity to $r=2m$ and from infinity to $r=4m$ in the field:

$$\vec{E}=\frac{10^5}{r}\hat{a}_r\ \ (V/m)\ ?$$

Solution:

$$W=-Q\int \vec{\mathbf{E}}\cdot\vec{\mathbf{dl}}=-Q\int\frac{10^5}{r}\hat{a}_r\cdot(dr\,\hat{a}_r+r\,d\theta\,\hat{a}_\theta+r\sin\theta\,d\phi\,\hat{a}_\phi)$$

$$W_2=-Q\int_\infty^2\frac{10^5}{r}dr=-Q\times10^5\times\left(\ln(r)\right)\Big|_\infty^2=-2\times10^{-9}\times10^5\,(0.693)$$

$$W_2=-1.38\times10^{-4}\ J$$

$$W_4=-Q\int_\infty^4\frac{10^5}{r}dr=-Q\times10^5\times\left(\ln(r)\right)\Big|_\infty^4=-2\times10^{-9}\times10^5\,(1.38)$$

$$W_4=-2.772\times10^{-4}\ J$$

$$\therefore\ W_4-W_2=-1.39\times10^{-4}\ J$$

Example(5): In spherical coordinate, point A is at a radius of $r_A = 2m$, while point B is at radius $r_B = 4m$. Given the field $\vec{E} = -\frac{16}{r^2}\hat{a}_r \ (V/m)$ find the potential at point A and B, assume zero reference at infinity and then find $V_A - V_B$? $Ans. \quad V_A = 2V_B = -8 \ Volt$

Solution:

$$V = -\int \mathbf{E} \cdot \overrightarrow{\mathbf{dl}} = -\int \left(-\frac{16}{r^2} \right) \hat{a}_r \cdot (dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi)$$

$$V_A = -\int_{\infty}^{2} \left(-\frac{16}{r^2} \right) dr = -\left(\frac{16}{r} \right)\Big|_{\infty}^{2} = -8\ V$$

$$V_B = -\int_{\infty}^{4} \left(-\frac{16}{r^2} \right) dr = -\left(\frac{16}{r} \right)\Big|_{\infty}^{4} = -4\ V$$

$$V_A - V_B = -8 - (-4) = -4V \qquad \Rightarrow\Rightarrow V_A = 2V_B = -8V$$

3-9: Gradient of Potential Difference:

From the defining equation for potential difference (V_{ab});, where (-$\vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}}$) is the differential potential difference over ($\overrightarrow{\mathbf{dl}}$ distance along the path from point (a) to point (b). Thus: $V_{ab}=-\int_a^b \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}}$

$$dV=-\vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}}=-\left|\vec{\mathbf{E}}\right|\left|\overrightarrow{\mathbf{dl}}\right|\cos\theta\ ------------(1)$$

$$\therefore\ \frac{\partial V}{\partial l}=-E\cos\theta\ ---------------(2)$$

If $\overrightarrow{\mathbf{dl}}=dl\,\hat{a}_E$ is in the same direction as $\vec{\mathbf{E}}$, $\theta=0$ *and hence* $\cos\theta=1$ *and* $\frac{\partial V}{\partial l}=-E$

since the potential decreases in the direction of ($\vec{\mathbf{E}}$) as indicated along the equipotential surfaces. Now, if $\overrightarrow{\mathbf{dl}}=dl\,(-\hat{a}_E)$ opposite to the direction of ($\vec{\mathbf{E}}$) then $\theta=\pi$ and hence:

$$\left.\frac{dV}{dl}\right|_{\overrightarrow{\mathbf{dl}}=dl\,(-\hat{a}_E)}=\left.\frac{dV}{dl}\right|_{Max.}=-E\cos\pi=E\ --------(3)$$

Thus, the **maximum derivative is in a direction opposite to** ($\vec{\mathbf{E}}$) and is equal to the magnitude of ($\vec{\mathbf{E}}$) When (V) is a function of (x, y, z), then the eq.(3) becomes:

$$\frac{\partial V}{\partial x} = -E_x \qquad \frac{\partial V}{\partial y} = -E_y \qquad \frac{\partial V}{\partial z} = -E_z$$

Since from the chin rule we have:

$$dV = \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}$$

$$dV = \left(\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \cdot \overrightarrow{\mathbf{dl}} = \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}$$

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\, grad\, V \text{ ----------- } (4)$$

$$\vec{\nabla} V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$$

Cartesian coordinate

$$\vec{\nabla} V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$

Cylindrical coordinate

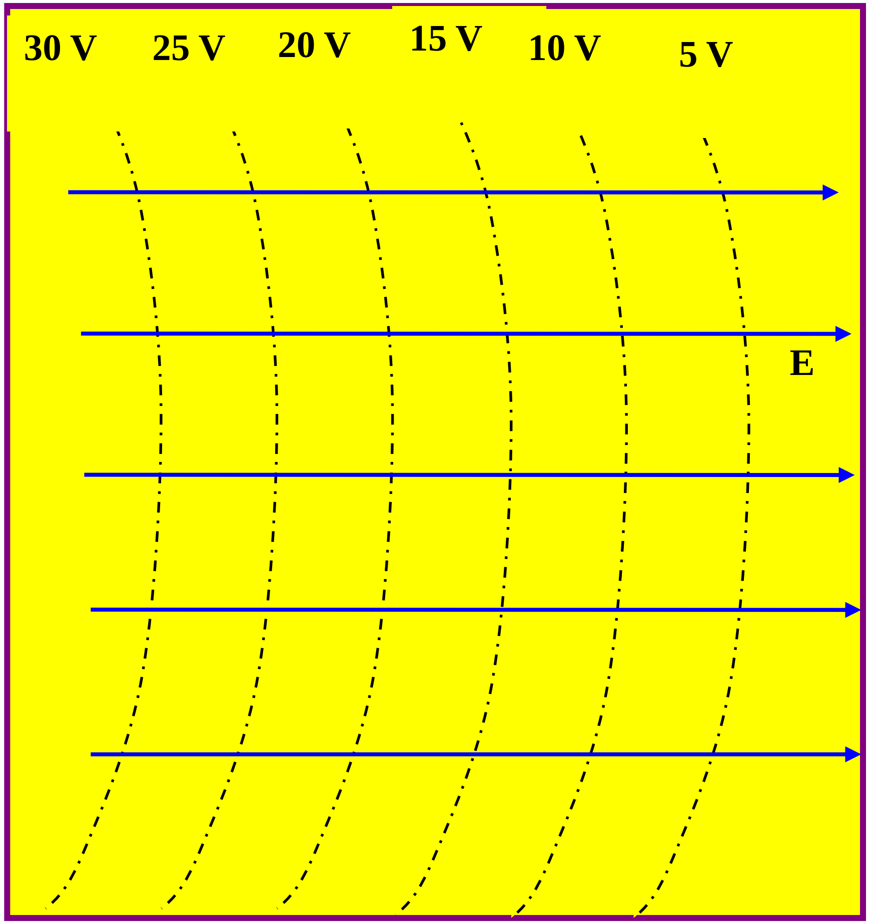
$$\vec{\nabla} V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$$

Spherical coordinate

Physically the (-) minus sign in eq.(4) indicates that the electric field intensity ($\vec{\mathbf{E}}$ is pointing in the direction of decreasing potential (V).

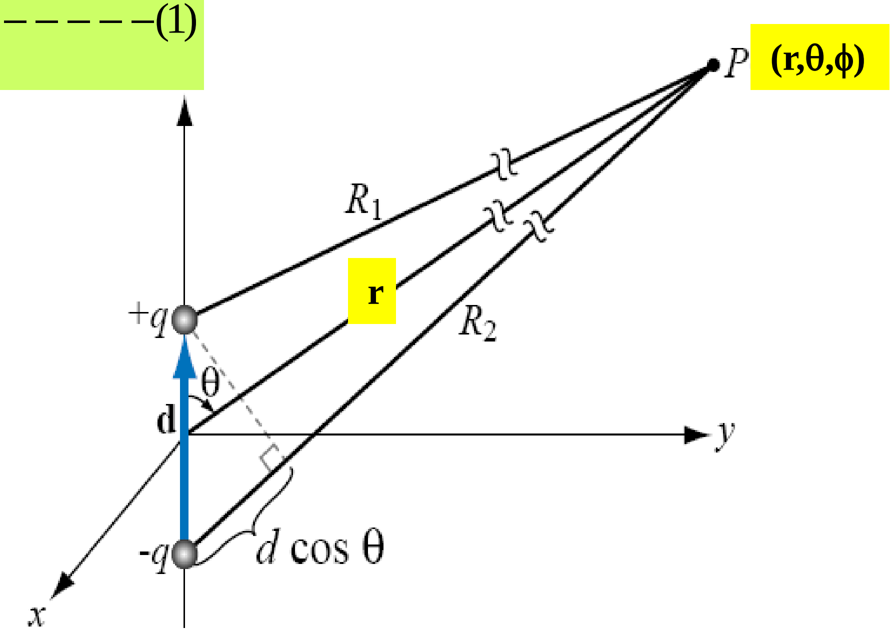
Or it means that the electric field intensity ($\vec{\mathbf{E}}$) is directed from the higher to lower levels of (V)

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\ gradV$$



Example(6): consider a pair of charge points of equal magnitude and opposite sign, **(+Q)** and **(-Q)** and separated by a small distance **(d)** as shown in figure below, such a pair termed an electric dipole. Find an expression for the electric field intensity and potential field of such a dipole at a distance **(r)** that is large compare to the charge separation **(d)**.

$$V = \frac{+Q}{4\pi\varepsilon R_1} - \frac{Q}{4\pi\varepsilon R_2} = \frac{Q}{4\pi\varepsilon}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \text{---------}(1)$$



From figure we see that: $R_2 - R_1 = d\cos\theta$

And because (r⟩⟩ d), then : $R_1 R_2 \cong r^2$

Therefore eq.(1) can be reduced to:

$$V_P = \frac{Q\,d\cos\theta}{4\pi\varepsilon r^2} \quad \text{------}(2)$$

$$V_P = \frac{\vec{\mathbf{P}}\cdot\hat{a}_r}{4\pi\varepsilon r^2} \qquad where \quad \vec{\mathbf{P}} = Q\vec{d} \qquad and \;\; \hat{a}_r \; is\ a\ unit\ vector$$

($\vec{\mathbf{P}} = Q\vec{d}$): The displacement of the electric charges due to an external electric field is called electric dipole moment.

$$\vec{\mathbf{E}} = -\vec{\nabla} V = \left(\frac{\partial}{\partial r}\,\hat{a}_r + \frac{1}{r}\,\frac{\partial}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \phi}\,\hat{a}_\phi \right) \left(\frac{Q\,d\cos\theta}{4\pi\,\varepsilon\,r^2} \right)$$

$$\vec{\mathbf{E}} = \frac{Q\,d\cos\theta}{2\pi\,\varepsilon\,r^3}\,\hat{a}_r + \frac{Q\,d\sin\theta}{2\pi\,\varepsilon\,r^3}\,\hat{a}_\theta + 0$$

Therefore, the electric field intensity for electric dipole is given by:

$$\vec{\mathbf{E}} = \frac{Q\,d}{2\pi\,\varepsilon\,r^3}\,(\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta) ------------(3)$$

(1). For point charge (monopole):

$$V = \frac{Q}{4\pi\,\varepsilon\,r}$$

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\,\varepsilon\,r^2}\,\hat{a}_r$$

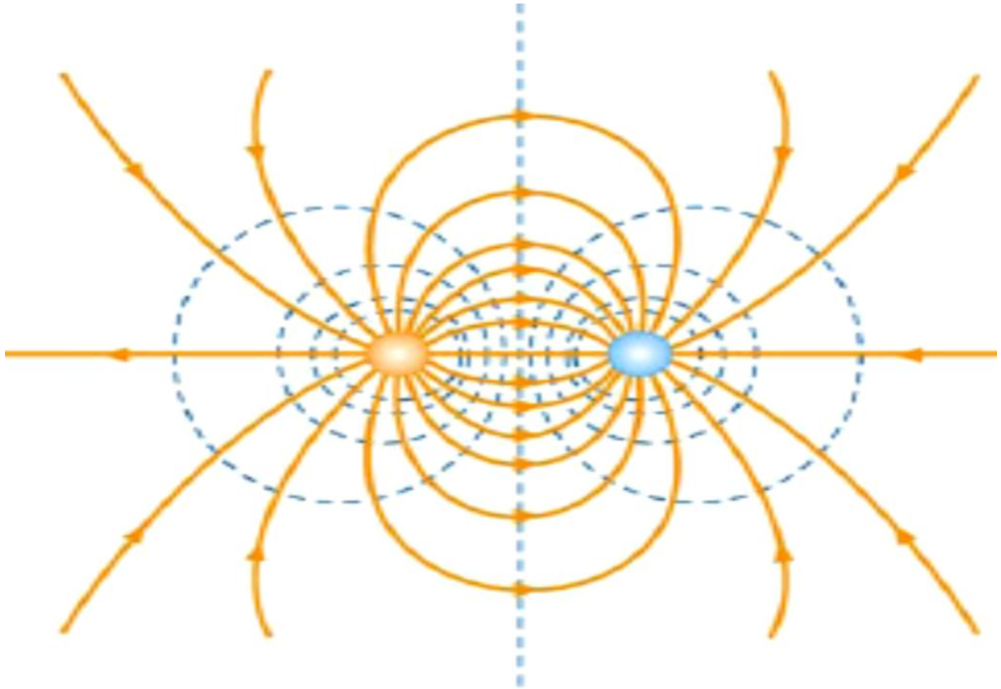
(2). For two charges (dipole) :

$$V = \frac{Q\,d\cos\theta}{4\pi\,\varepsilon\,r^2}$$

$$\vec{\mathbf{E}} = \frac{Q\,d}{2\pi\,\varepsilon\,r^3}\,(\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)$$

From these two set of equation we notice that the ($\vec{E}$) varies inversely with (r^2) and (V) varies inversely with (r) for monopole. While for dipole ($\vec{E}$) varies inversely with (r^3) and (V) varies inversely with (r^2).

Therefore, the electric field intensity ($\vec{E}$) due to successive higher order multipoles [such as quadrapole (two dipole) or octapole (two qudrapole)] are vary inversely as r^4, r^5, r^6….. and (V) vary inversely with r^3, r^4, r^5 …….



$$\vec{\mathbf{E}} = -\vec{\nabla} V \qquad \vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}} \qquad \vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_v \qquad Q = \int \rho_v \, dv$$

Example(7): **In the potential field** $V = 5r^2$ **, how much charge is located within a unit sphere centered at the origin ?**

Solution:

The electric field intensity is related to the electric potential through the gradient equation as:

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\left(\frac{\partial V}{\partial r}\,\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\hat{a}_\phi\right) \quad \Rightarrow\Rightarrow \vec{\mathbf{E}} = -10\,r\,\hat{a}_r$$

$$\vec{\mathbf{D}} = \varepsilon_\circ \vec{\mathbf{E}} = -10\,\varepsilon_\circ\, r\,\hat{a}_r \quad \textit{and} \quad \rho_v = \vec{\nabla}\cdot\vec{\mathbf{D}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\mathbf{D}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\mathbf{D}_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \mathbf{D}_\phi}{\partial\phi}$$

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(-10\,\varepsilon_\circ\, r)\right) + 0 + 0 \quad \Rightarrow\Rightarrow \rho_v = -30\,\varepsilon_\circ$$

$$Q = \int_v \rho_v\, dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}(-30\,\varepsilon_\circ)\, r^2 \sin\theta\; dr\; d\theta\; d\phi = -30\,\varepsilon_\circ \int_0^{2\pi}\int_0^{\pi} \sin\theta\; d\theta\; d\phi \int_0^1 r^2\, dr$$

$$Q = -30\,\varepsilon_\circ \times (4\pi) \times \frac{r^3}{3}\Big|_0^1 \Rightarrow\Rightarrow\; Q = -40\,\varepsilon_\circ\,\pi$$

Example(8): A disk of $0 \le \rho \le a \quad , \quad z=0 \quad and \quad 0 \le \phi \le 2\pi$ **, carries a surface charge density of ,** $\rho_s = \rho_\circ \frac{\rho^2}{a^2} \, (C/m^2)$ **find** $V \quad at \quad (0,0,z)$ **in free space?**

Solution:

For a disk or sheet of charges the electric potential field is related to charge density by the following equation:

$$V = \frac{1}{4\pi\varepsilon_\circ} \int \frac{\rho_s \, ds}{R} ------------(1)$$

$$\vec{\mathbf{R}} = (0-\rho)\hat{a}_\rho + (z-0)\hat{a}_z = -\rho\hat{a}_\rho + z\hat{a}_z$$

$$R = \left|\vec{\mathbf{R}}\right| = \sqrt{\rho^2 + z^2} \quad --------(2) \quad and \quad ds = \rho \, d\rho \, d\phi ---------(3)$$

$$V = \frac{1}{4\pi\varepsilon_\circ} \int \frac{\rho_\circ \frac{\rho^2}{a^2} \rho \, d\rho \, d\phi}{\sqrt{\rho^2 + z^2}} = \frac{1}{4\pi\varepsilon_\circ a^2} \int_0^{2\pi} d\phi \int_0^a \frac{\rho^3 \, d\rho}{\sqrt{\rho^2 + z^2}}$$

$$Let \quad \rho = z\tan\theta$$

$$d\rho = z\sec^2$$

$$V = \frac{1}{2\varepsilon_\circ a^2} \int_0^a \frac{z^4 \tan^3\theta \sec\theta \, d\theta}{z\sec\theta} = \frac{z^3}{2\varepsilon_\circ a^2} \int_0^a \tan\theta (1+\sec^2\theta) \, d\theta$$

$$V = \frac{z^3}{2\varepsilon_\circ a^2}\left[\int_0^a \tan\theta \, d\theta + \int_0^a \tan\theta \sec^2\theta \, d\theta\right] ------------(4)$$

$$V = \frac{z^3}{2\varepsilon_\circ a^2}\left(-\ln(\cos\theta) + \tan^2\theta)\right|_0^a$$

since:

$$\tan\theta = \frac{\rho}{z} \qquad then \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}}$$

$$and \quad \tan^2\theta = 1 + \sec^2\theta$$

Substituting these identity into eq.(4) we get:

$$V = \frac{z^3}{2\varepsilon_\circ a^2}\left[-\ln\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right) + \left(\frac{\rho^2}{z^2}\right)\right]\Bigg|_0^a$$

$$V = \frac{z^3}{2\varepsilon_\circ a^2}\left[-\ln\left(\frac{z}{\sqrt{a^2 + z^2}}\right) + \left(\frac{a^2}{z^2}\right) + \ln\left(\frac{z}{\sqrt{0 + z^2}}\right) - \left(\frac{0}{z^2}\right)\right]$$

$$V = \frac{z^3}{2\varepsilon_\circ a^2}\left[\left(\frac{a^2}{z^2}\right) - \ln\left(\frac{z}{\sqrt{a^2 + z^2}}\right)\right]$$

Home Work

Q_1/ Find $\vec{E}, \vec{D}, \rho_v, and\ \ Q$ for each of the following potential difference fields:

a- $V = 7x^2$ **b-** $V = 2\rho\cos\phi$ **c-** $V = \frac{5}{r}\sin\theta$

Q_2/ For each of these potential fields, find $V, \vec{E}, \vec{D}, and\ \ \rho_v$ at (2,-2,2):

a- $V = 3xy + z + 4$ **b-** $V = 5\sin\phi\ e^{-\rho+z}$ **c-** $V = \frac{4}{r}\sin\theta\ \sin\phi$

Q_3/ How much charge must be located within a unit sphere centered at the origin in order to produce the potential field $V = \frac{-6r^5}{\varepsilon_\circ}$, for $r \le 1\ m$?

Q_4/ A circular disk of radius (a) carries charge density $\rho_s = \frac{1}{\rho}\ (C/m^2)$. Calculate the electric potential at $(0,0,h)$?

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Q_5/ A sheet of charge density $\rho_s = 100\ \ (nC/m^2)$ occupies the x-z-plane at y=0. Find the work required to move a $Q = 2\ nC$ charge from $A(-5,10,2)m\ \ to\ \ \ B(2,3,0)m$, then evaluate electric potential difference V_{AB} ?

Q_6/ For the electrostatic dipole moment, given that $d = 1cm\ \ and\ \ |E| = 2\ (mv/m)\ \ at\ \ \ r = 1m\ \ and\ \ \theta = 0^\circ$. Find $\vec{E}\ \ at\ \ r = 2m\ \ \ and\ \ \ \theta = 90^\circ$?

Q_7/ Find the work done in moving a point charge $Q = -20\,\mu C$ from the origin to $(4,2,0)m$ in the field:

$\vec{E} = 2(x+4y)\hat{a}_x + 8x\hat{a}_y \ (N/C)$ along the path $8y = x^2$? Ans. (1.6 mJ)

Q_8/ Given the cylindrical coordinate electric fields as: $\vec{E} = \frac{5}{\rho}\hat{a}_\rho \ (V/m) \quad for \quad 0 \le \rho \le 2m$ and

$\vec{E} = 2.5\,\hat{a}_\rho \ (V/m) \quad for \quad \rho > 2m$. Find the potential difference V_{AB} for $A(1,0,0)$ *and* $B(4,0,0)$? $Ans. 8.47\ V$

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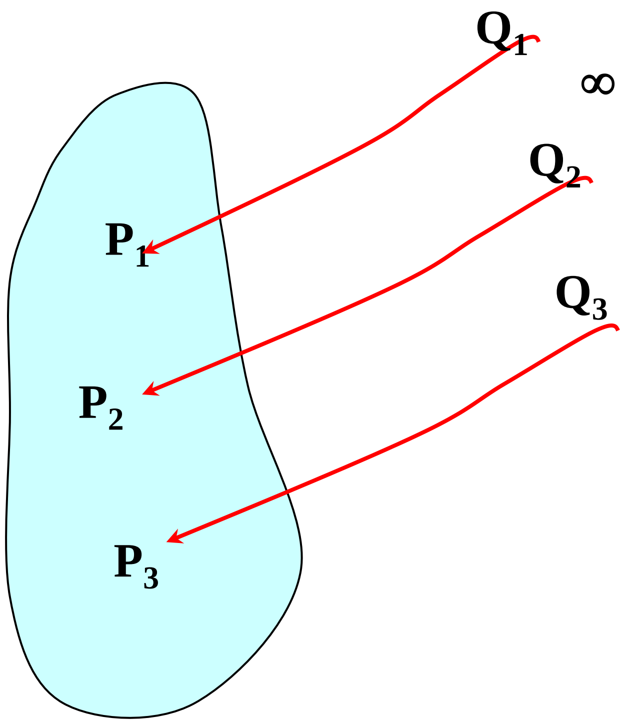
3-10: Energy and Energy Density in Electrostatic Field:

To determine the energy present in an assembly of charges, we must first determine the work necessary to assemble them. Suppose we wish to position three point charges $Q_1, Q_2 \ and \ Q_3$ in an initially empty space as shown in figure.

(1). No work is required to transfer (Q_1) from infinity to (P_1), because the space is initially charge free and there is no electric ($\vec{\mathbf{E}} = 0$) field and hence:

$$W1 = -Q \int \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 \text{ -----------------} (1)$$



(2).The work done in transferring (Q_2) from infinity to (P_2) is given by:

$$W2 = Q_2 V_{21} ------(2) \qquad where,\ V_{21}\ means\ \ V_2\ \ due\ to\ \ (Q_1)$$

(3)Similarly, the work done in positioning (Q_3) at (P_3) is equal to :

$$W3 = Q_3 (V_{31} + V_{32}) \ --------(3)$$

Therefore, the total work done in positioning the three charges is:

$$W_E = W1 + W2 + W3 = \quad 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) -------(4)$$

Now, if the charges were positioned in reverse order; then the total work is given by

$$W_E = W1 + W2 + W3 = \quad Q_1 (V_{12} + V_{13}) + Q_2 V_{23} + 0 -------------(5)$$

Adding eqs.(4) with (5) we get:

$$2W_E = Q_1(V_{12} + V_{13}) + Q_1(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$$

$$W_E = \frac{1}{2}Q_1 V_1 + \frac{1}{2}Q_2 V_2 + \frac{1}{2}Q_3 V_3 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3) --------(6)$$

Where, (V_1 , V_2 and V_3) are total potential at points (P_1, P_2 and P_3), respectively. In general, if there are (N) point charges, then eq.(6) can be written as:

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \quad (in\ Joules\ ) ----------(7)$$

However, if instead of point charges, the region has a continuous charge distribution, the summation in eq.(7) becomes integration; that is:

$$W_E = \frac{1}{2}\int V \rho_l \, dl \ ----------(8) \qquad Line\ charge\ distribution$$

$$W_E = \frac{1}{2}\int V \rho_s \, ds \ ----------(9) \qquad Surface\ charge\ distribution$$

$$W_E = \frac{1}{2}\int V \rho_v \, dv \ ----------(10) \qquad Volume\ charge\ distribution$$

Since, $\rho_v = \vec{\nabla} \cdot \vec{\mathbf{D}}$, so eq.(10) can be further developed to yield:

$$W_E = \frac{1}{2}\int V\,(\vec{\nabla} \cdot \vec{\mathbf{D}})\; dv \;--------(11)$$

But for any vector $\vec{\mathbf{A}}$ and scalar V , we have the following identity:

$$\vec{\nabla} \cdot V\,\vec{\mathbf{A}} = \vec{\mathbf{A}} \cdot \vec{\nabla} V + V\,(\vec{\nabla} \cdot \vec{\mathbf{A}}) \;\Rightarrow \textit{and hence} \quad V\,(\vec{\nabla} \cdot \vec{\mathbf{A}}) = \vec{\nabla} \cdot V\,\vec{\mathbf{A}} - \vec{\mathbf{A}} \cdot \vec{\nabla} V \;------(12)$$

Applying the identity given in eq.(12) on to eq.(11) we get:

$$W_E = \frac{1}{2}\int (\vec{\nabla} \cdot V\,\vec{\mathbf{D}})\, dv - \frac{1}{2}\int \vec{\mathbf{D}} \cdot \vec{\nabla} V\, dv \quad ---------(13)$$

And according to divergence theorem we can rewrite the eq.(13) as given below:

$$W_E = \frac{1}{2}\int V\,\vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} - \frac{1}{2}\int \vec{\mathbf{D}} \cdot \vec{\nabla} V\; dv \;-----------(14)$$

Since (V) varies inversely as (r) and (D) varies inversely as (r^2) for point charges; and (V) varies inversely as (r^2) and (D) varies inversely as (r^3) for dipoles and so on. Hence, $(V\,\vec{D})$ in the first term of the right hand side of eq.(14) must vary inversely at least as (r^3) while (ds) varies directly as (r^2). Consequently, the first integral in eq.(14) must tend to zero as the surface (S) becomes large. Thus, eq.(14) reduces to:

$$W_E = -\frac{1}{2}\int \vec{\mathbf{D}}\cdot\vec{\nabla} V\, dv \quad , \quad but \quad \vec{\mathbf{E}} = -\vec{\nabla} V \quad then$$

$$W_E = \frac{1}{2}\int \vec{\mathbf{D}}\cdot\vec{\mathbf{E}}\; dv \qquad and \quad \vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}} \qquad hence$$

$$W_E = \frac{1}{2}\,\varepsilon \int \vec{\mathbf{E}}\cdot\vec{\mathbf{E}}\; dv$$

Therefore, the total energy stored in electrostatic field can be finally simplified to:

$$W_E = \frac{1}{2}\,\varepsilon \int E^2\; dv \qquad (in\;\; Joule) ------------(15)$$

For this we can define electrostatic energy density $w_E \;\; in\; (\,Joule/m^3)$ as:

$$w_E = \frac{dW}{dv} = \frac{1}{2}\varepsilon\; E^2 = \frac{1}{2}\,\frac{D^2}{\varepsilon} -------------(16)$$

Hence we can say that the electrostatic energy is given by:

$$W_E = \int w_e\, dv \qquad ----(17)$$

Example(9):Given $\vec{E} = 5xy\,\hat{a}_x + 3z\,\hat{a}_z \ (V/m)$

, find the electrostatic potential energy stored in a volume defined by:

$0 \le x \le 2m \ , \ 0 \le y \le 1\,m \quad and \quad 0 \le z \le 1\,m \quad assume\ \varepsilon = \varepsilon_\circ$

Solution:

The electrostatic energy stored in this field bounded by these regions is calculated as follows:

$$w_E = \frac{1}{2}\varepsilon_\circ \int_v \mathbf{E}^2\,dv = \frac{1}{2}\varepsilon_\circ \int_0^1\int_0^1\int_0^2 \left[25x^2y^2 + 9z^2\right] dx\,dy\,dz$$

$$w_E = \frac{1}{2}\varepsilon_\circ \left[25\left(\frac{x^3}{3}\right)\Big|_0^2 \left(\frac{y^3}{3}\right)\Big|_0^1 (z)\Big|_0^1 + 9\left(\frac{z^3}{3}\right)\Big|_0^1 (x)\Big|_0^2 (y)\Big|_0^1\right]$$

$$w_E = \frac{1}{2}\varepsilon_\circ \left[\frac{25}{9}\times 8\times 1\times 1 + \frac{9}{3}\times 1\times 2\times 1\right]$$

$$w_E = \frac{1}{2}\varepsilon_\circ \left[\frac{200}{9} + \frac{18}{3}\right] = \frac{1}{2}\varepsilon_\circ \left[\frac{200+54}{9}\right] = \frac{1}{2}\varepsilon_\circ \times \frac{254}{9}$$

$$w_E = 0.124\ nJ$$

3-12: Derivative of Poisson's and Laplace's Equations:

To derive the Poisson's and Laplace's equation which are useful for determining the electrostatic potential (V) in regions at whose boundaries (V) is known. The following steps give these two equations:

With ($\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$), the differential form of Gauss's law given by: ($\vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_v$) may be written as:

$$\vec{\nabla} \cdot \varepsilon \vec{\mathbf{E}} = \rho_v \quad ------(1) \qquad but \quad we\ have \quad \vec{\mathbf{E}} = -\vec{\nabla} V -------(2)$$

Thus, $$\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho_v}{\varepsilon} \qquad \Rightarrow \qquad \nabla^2 V = -\frac{\rho_v}{\varepsilon} ------------(3)$$

Equation(3) is called Poisson's equation for a volume (v) containing a volume charge density (ρ_v), the solution for (V) in eq.(3) is given by:

$$V = \frac{1}{4\pi\varepsilon} \int_v \frac{\rho_v \, dv}{R} ----------(4)$$

If the system or medium under consideration contains no free charges ($\rho_v = 0$), then eq.(3) reduce to:

$$\nabla^2 V = 0 -------------(5)$$

Equation (5) is called Laplace's equation and in Cartesian, cylindrical and spherical coordinates is given by:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad ----------(6) \quad \textit{cartesian coordinate}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \quad ------(7) \quad \textit{cylindrical coordinate}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad ----(8) \quad \textit{spherical coordinate}$$

Example(10) A potential field is described by, $V = 5x^2 \ (Volt)$ is Laplace's equation satisfied? If not , find the charge density in the region?

Solution:

In order to satisfy Laplace's equation for given function, it must be: $\nabla^2 V = 0$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{d^2(5x^2)}{dx^2} = 10$$

Therefore this function dose not satisfies Laplace's equation. Hence, to determine the charge density for this potential function, the following step must be done:

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\left(\frac{\partial V}{\partial x}\,\hat{a}_x + \frac{\partial V}{\partial y}\,\hat{a}_y + \frac{\partial V}{\partial z}\,\hat{a}_z\right) \quad \Rightarrow\Rightarrow \vec{\mathbf{E}} = -10\,x\,\hat{a}_x$$

$$\vec{\mathbf{D}} = \varepsilon_\circ \vec{\mathbf{E}} = -10\,x\,\varepsilon_\circ\,\hat{a}_x \quad and \qquad \rho_v = \vec{\nabla} \cdot \vec{\mathbf{D}} = \frac{\partial \mathbf{D}_x}{\partial x} + \frac{\partial \mathbf{D}_y}{\partial y} + \frac{\partial \mathbf{D}_z}{\partial z}$$

$$\rho_v = -10\,\varepsilon_\circ + 0 + 0 \qquad \Rightarrow\Rightarrow \rho_v = -10\,\varepsilon_\circ$$

Or it can be obtain the same result from the application of Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_\circ} \quad and \quad from\ \ Laplace's\ equation\ \ we\ have\ \nabla^2 V = 10\ then$$

$$10 = -\frac{\rho_v}{\varepsilon_\circ} \quad \Rightarrow\Rightarrow \quad \rho_v = -10\,\varepsilon_\circ$$

Home Work

Q$_1$/ In cylindrical coordinates, equation: $\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+10=0$ is called:

a. **Laplace's equatuion** b. **Helmholtiz Equation** c. **Poisson's Equation** d. **Maxwell's Equation**

Q2/ Which of the following potential does not satisfy Laplace's equation:

a. $V=2x+5$ **b.** $V=10\ x\ y$ **c.** $V=r\ \cos\theta$ **d.** $V=\frac{10}{r}$ **e.** $V=\rho\cos\phi+10$

Q3/ Solve the poison's equation in free space and in cartesian coordinates for

$0\le x\le 1\ m\ \ if\ \ \rho_v=10^{-9}\ (2+\sin\pi x)\ (C/m^3)\ \ and\ \ V=0\ \ at\ \ x=0\ \ and\ \ x=1$ also find $V\ \ at\ \ x=\frac{1}{2}$

Q$_4$/ Let $V=A\,x^2\,y\,z+B\ \ (Volt)$, find $A\ \ and\ \ B$ if : **a.** $V=0\ \ at\ the\ origin\ \ and\ \ V=100\ volt\ \ at\ \ (2,-1,5)$

?

b. $V=0\ \ at\ the\ origin\ \ and\ \ |E|=20\ (V/m)\ \ at\ \ (2,-1,5)$ **c.** $V=100\ \ (Volt)\ \ at\ the\ origin\ \ and\ \ V=0\ \ at\ \ (2,-1,5)$

Q$_5$/ For the following potential functions, use the gradient equation to find the $\vec{E}\ ,\ \vec{D}\ \ and\ \ \rho_v$ in free space:

a. $V=x+y^2\ z$ **b.** $V=\rho^2\ \sin\phi$ **c.** $V=r\ \sin\theta\ \cos\phi$