Chapter Four /// Part (III)

Electrostatic Field in Material Space & Boundary Conditions

4-8: Linear, Isotropic and Homogenous Dielectrics:

The dielectric material mediums are classified as linear, isotropic and homogenous according to the following properties:

(1). A dielectric material is said to be linear, when, at a given point, (ε) is a constant. Thus a plot of $|\vec{\mathbf{D}}|\ versus\ |\vec{\mathbf{E}}|$ will be a straight line. If the permittivity, at a given point is a function of $|\vec{\mathbf{E}}|,\ \varepsilon = \varepsilon(|\vec{\mathbf{E}}|)$, then the material is non-linear and a plot of $|\vec{\mathbf{D}}|\ versus\ |\vec{\mathbf{E}}|$ will not be a straight line.

(2). A dielectric material is said to be homogenous when (ε) does not vary from point to another; thus (ε)is not a function of position. If the permittivity varies with position [$\varepsilon = \varepsilon(x, y, z)$], then the material is called non-homogenous.

(3). **The dielectric material is said to be Isotropic, when ($\vec{\mathbf{P}}, \vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$) are in the same direction, that is:** $\mathbf{P}_x = \varepsilon_\circ \chi_e \mathbf{E}_x$, $\mathbf{P}_y = \varepsilon_\circ \chi_e \mathbf{E}_y$ $\mathbf{P}_z = \varepsilon_\circ \chi_e \mathbf{E}_z$ **and** $\mathbf{D}_x = \varepsilon_\circ \mathbf{E}_x$ $\mathbf{D}_y = \varepsilon_\circ \mathbf{E}_y$ **and** $\mathbf{D}_z = \varepsilon_\circ \mathbf{E}_z$ **. A non-isotropic dielectric materials are those in which the** $\vec{\mathbf{P}}, \vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ **are not in the same direction, while have a tensor extension as given below:**

$$\mathbf{P}_x = \chi_{11} \varepsilon_\circ \mathbf{E}_x + \chi_{12} \varepsilon_\circ \mathbf{E}_y + \chi_{13} \varepsilon_\circ \mathbf{E}_z$$
$$\mathbf{P}_y = \chi_{21} \varepsilon_\circ \mathbf{E}_x + \chi_{22} \varepsilon_\circ \mathbf{E}_y + \chi_{23} \varepsilon_\circ \mathbf{E}_z$$
$$\mathbf{P}_z = \chi_{31} \varepsilon_\circ \mathbf{E}_x + \chi_{32} \varepsilon_\circ \mathbf{E}_y + \chi_{33} \varepsilon_\circ \mathbf{E}_z$$

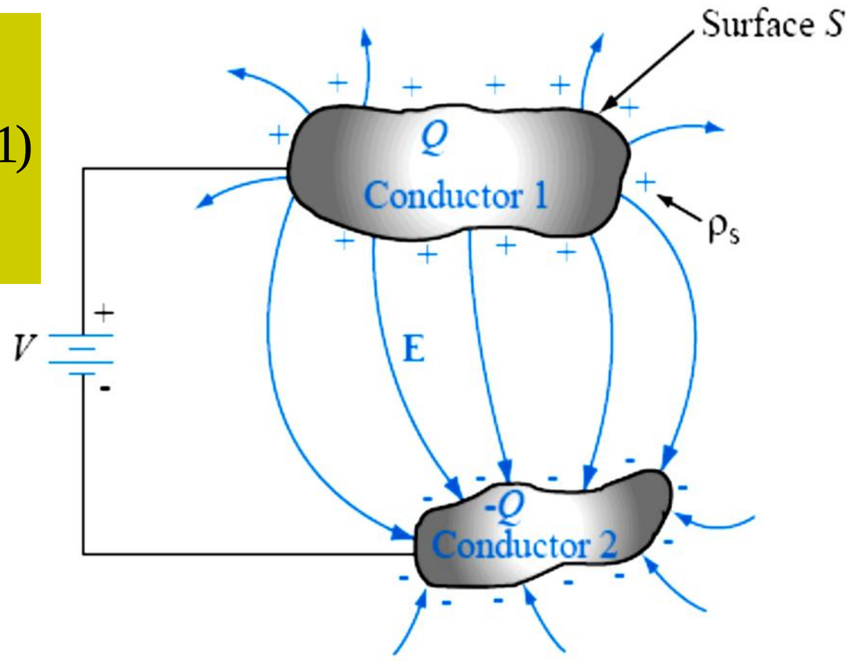
$$\mathbf{D}_x = \varepsilon_{11} \mathbf{E}_x + \varepsilon_{12} \mathbf{E}_y + \varepsilon_{13} \mathbf{E}_z$$
$$\mathbf{D}_y = \varepsilon_{21} \mathbf{E}_x + \varepsilon_{22} \mathbf{E}_y + \varepsilon_{23} \mathbf{E}_z$$
$$\mathbf{D}_z = \varepsilon_{31} \mathbf{E}_x + \varepsilon_{32} \mathbf{E}_y + \varepsilon_{33} \mathbf{E}_z$$

$$\begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \\ \mathbf{D}_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} \mathbf{E}_x \\ \mathbf{E}_y \\ \mathbf{E}_z \end{bmatrix}$$

4-9: Capacitance (C):

The capacitance of a capacitor is defined as the ratio of the magnitude of the charge (Q) on one of the plates to the potential difference (V) between them; that is:

$$\mathbf{C} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\oint_S \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}} = \frac{\varepsilon_\circ \oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}} \quad --------(1)$$



Generally, to have a capacitor we must have two or (more) conductors carrying equal but opposite charges. The conductors are some times referred to us the plates of the capacitor. The plates may be separated by free space or by a dielectric material.

From eq.(1), if we have a parallel plates of uniform cross sectional area (A) and separated by (d), the capacitance is now given by:

$$\mathbf{C} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\varepsilon_\circ \oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}} = \frac{\varepsilon_\circ \, \mathbf{E} \, A}{\mathbf{E} \, d} \Rightarrow\Rightarrow \quad and \quad hence, \; \mathbf{C} = \frac{\varepsilon_\circ \; A}{d} --------(2)$$

Also we have the resistance (R) of a plate of conductor given by:

$$\mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\int \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}}{\sigma \oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{d}\,s}} -----------------(3)$$

Comparing eq.(1) with eq.(3) we obtain a relationship between resistance and capacitance as given below:

$$\mathbf{R}\,\sigma = \frac{\varepsilon_\circ}{\mathbf{C}} \Rightarrow\Rightarrow \quad \mathbf{R}\,\mathbf{C} = \frac{\varepsilon_\circ}{\sigma} ----------(4)$$

The capacitance (C) is a physical property of the capacitor and it measured in Farad (F). Employing eq.(1), the capacitance of any given tow parallel arbitrary plate shapes can be obtained, using either one of the following methods:

$$\mathbf{C} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\oint_S \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}} = \frac{\varepsilon_\circ \oint_S \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}} \text{ --------- } (1)$$

(1). Assuming (Q) and determining (V) in terms of (Q) [Gauss's law]

$$\mathbf{Q} = \varepsilon \oint_s \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}$$

(2). Assuming (V) and determining (Q) in terms of (V) [Laplace's equation]

$$\nabla^2 V = 0$$

Example (5):

(a). show that the capacitance of Figure(a) is given by: $\mathbf{C} = \frac{\varepsilon_{r1}\,\varepsilon_\circ\,A_1}{d_1} + \frac{\varepsilon_{r2}\,\varepsilon_\circ\,A_2}{d_2} = \mathbf{C}_1 + \mathbf{C}_2$

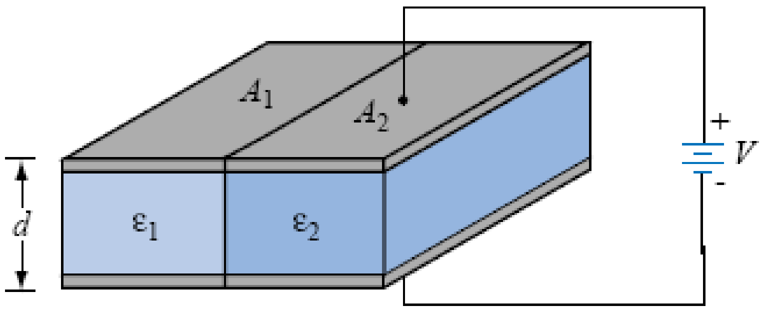
(b). show that the capacitance of Figure(b) is given by: $\frac{1}{C} = \frac{d_1}{\varepsilon_{r1}\,\varepsilon_\circ\,A_1} + \frac{d_1}{\varepsilon_{r1}\,\varepsilon_\circ\,A_1} = \frac{1}{\mathbf{C}_1} + \frac{1}{\mathbf{C}_2}$

Solution:

(a). because the (V) is common for both dielectrics, then:

$$\vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_2 = \frac{\mathbf{V}}{d}\,\hat{a}_n \quad \text{since,} \quad \vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}} = \varepsilon_\circ\,\varepsilon_r\,\vec{\mathbf{E}}$$

hence, $$\frac{\vec{\mathbf{D}}_1}{\varepsilon_\circ\,\varepsilon_{r1}} = \frac{\vec{\mathbf{D}}_2}{\varepsilon_\circ\,\varepsilon_{r2}} = \frac{\mathbf{V}}{d}\,\hat{a}_n$$



(a)

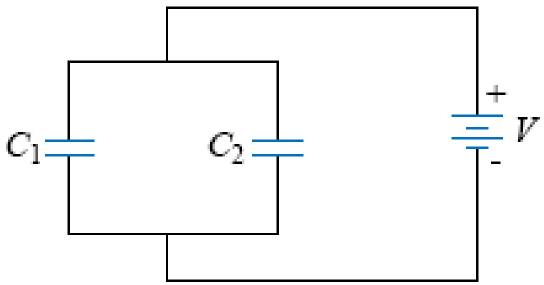
Also we have: $$\mathbf{D}_n = \rho_s = \frac{\mathbf{Q}}{\mathbf{A}} \quad and \quad \mathbf{D} = \varepsilon_\circ\,\varepsilon_r\,\mathbf{E}$$

Hence, $$\rho_{s1} = \varepsilon_\circ\,\varepsilon_r\,\mathbf{E}_1 = \varepsilon_\circ\,\varepsilon_{r1}\frac{\mathbf{V}}{d} \qquad and \qquad \rho_{s2} = \varepsilon_\circ\,\varepsilon_r\,\mathbf{E}_2 = \varepsilon_\circ\,\varepsilon_{r2}\frac{\mathbf{V}}{d}$$

The total charge: $$\mathbf{Q} = \rho_{s1}\,\mathbf{A}_1 + \rho_{s2}\,\mathbf{A}_2 = \frac{\mathbf{V}\,\varepsilon_\circ}{d}\left(\mathbf{A}_1\,\varepsilon_{r1} + \mathbf{A}_1\,\varepsilon_{r2}\right)$$

Therefore, the total capacitance is now given by:

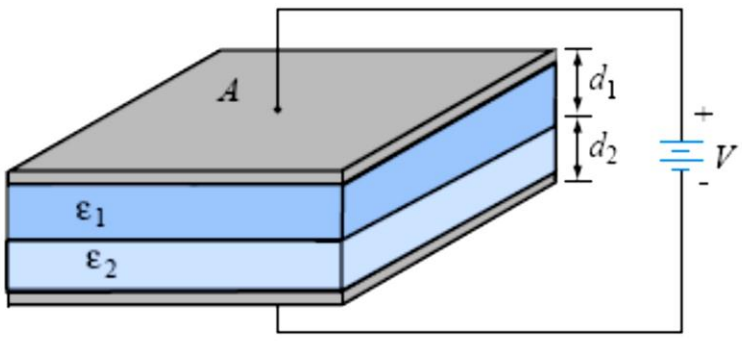
$$\mathbf{C} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\mathbf{A}_1\,\varepsilon_{r1}\,\varepsilon_\circ}{d} + \frac{\mathbf{A}_2\,\varepsilon_{r2}\,\varepsilon_\circ}{d} = \mathbf{C}_1 + \mathbf{C}_2$$



(b)

(a). because the (D) is common for both dielectrics, and is given by:

$$\vec{\mathbf{D}}_n = \frac{\mathbf{Q}}{\mathbf{A}}\hat{a}_n \quad and \quad \vec{\mathbf{E}} = \frac{\vec{\mathbf{D}}}{\varepsilon}$$, Therefore,



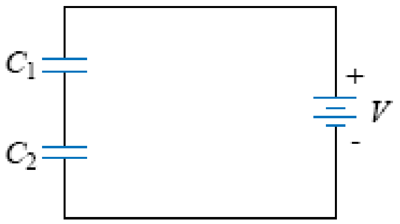
(a)

$$\vec{\mathbf{E}}_1 = \frac{\mathbf{Q}}{\mathbf{A}\varepsilon_\circ \varepsilon_{r1}}\hat{a}_n \quad and \quad \vec{\mathbf{E}}_2 = \frac{\mathbf{Q}}{\mathbf{A}\varepsilon_\circ \varepsilon_{r2}}\hat{a}_n$$

Since, $\mathbf{V} = \mathbf{E}\,d$ hence, $\mathbf{V}_1 = \vec{\mathbf{E}}_1 d_1 = \frac{\mathbf{Q}\ d_1}{\mathbf{A}\varepsilon_\circ \varepsilon_{r1}}$ and $\mathbf{V}_2 = \vec{\mathbf{E}}_2 d_2 = \frac{\mathbf{Q}\ d_2}{\mathbf{A}\varepsilon_\circ \varepsilon_{r2}}$

But the total voltage across the two capacitance is given by:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = \mathbf{Q}\left(\frac{d_1}{\mathbf{A}\varepsilon_\circ \varepsilon_{r1}} + \frac{d_2}{\mathbf{A}\varepsilon_\circ \varepsilon_{r2}}\right)$$



(b)

Therefore, the total capacitance now is given by:

$$\frac{1}{\mathbf{C}} = \frac{\mathbf{V}}{\mathbf{Q}} = \left(\frac{d_1}{\mathbf{A}\varepsilon_\circ \varepsilon_{r1}} + \frac{d_2}{\mathbf{A}\varepsilon_\circ \varepsilon_{r2}}\right) = \frac{1}{\mathbf{C}_1} + \frac{1}{\mathbf{C}_2}$$

Example (6): **Find the capacitance and electrostatic energy stored between the parallel plates of area (A) and the plates are separated by a distance (d) and filled with a dielectric material of permittivity (ε)**

Solution:

From the mathematical definition of the uniform cross sectional area of a plate of capacitance we have:

$$\mathbf{C}=\frac{\mathbf{Q}}{\mathbf{V}}=\frac{\varepsilon\oint_{S}\vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{ds}}}{\int_{L}\vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}}}=\frac{\varepsilon_{\circ}\,\varepsilon_{r}\,\mathbf{E}\,A}{\mathbf{E}\,d}$$

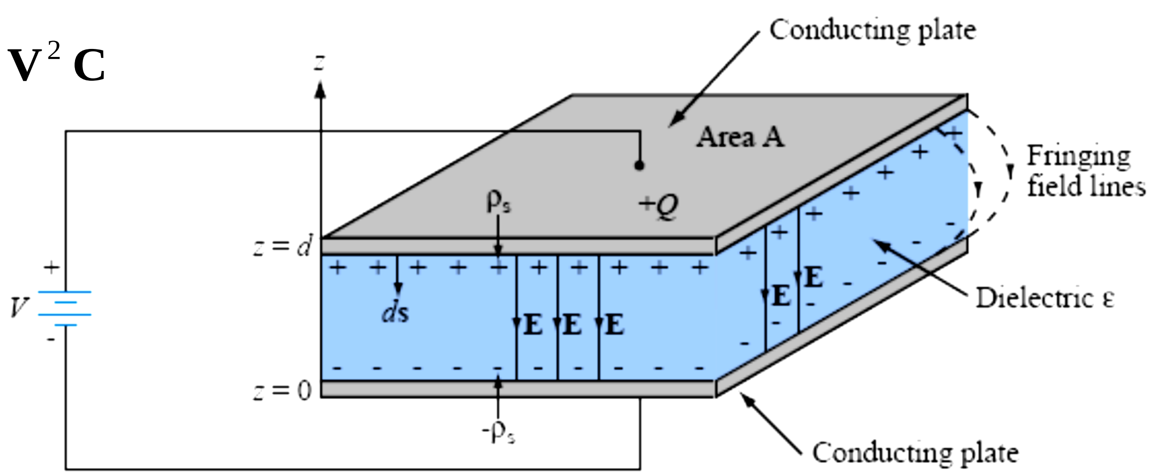
hence

$$\mathbf{C}=\frac{\varepsilon_{\circ}\,\varepsilon_{r}\,A}{d}$$

The electrostatic energy stored in a region of charges is calculated through a following relation:

$$w_{E}=\frac{1}{2}\int_{v}\vec{\mathbf{D}}\cdot\vec{\mathbf{E}}\;dv=\frac{1}{2}\int_{v}\frac{\mathbf{Q}}{A}\cdot\frac{\mathbf{V}}{\mathbf{d}}\;dv\quad,\quad \sin ce\quad dv=(\mathbf{A}\,\mathbf{d})$$

$$w_{E}=\frac{1}{2}\int_{v}\frac{\mathbf{Q}}{A}\cdot\frac{\mathbf{V}}{\mathbf{d}}\;dv=\frac{1}{2}\,\mathbf{Q}\,\mathbf{V}=\frac{1}{2}\,\frac{\mathbf{Q}^{2}}{\mathbf{C}}=\frac{1}{2}\,\mathbf{V}^{2}\,\mathbf{C}$$



Or in another way, the electrostatic energy stored in this region is calculated as:

$$w_E = \frac{1}{2}\int_v \vec{\mathbf{D}}\cdot\vec{\mathbf{E}}\,dv = \frac{1}{2}\,\varepsilon\int_v \mathbf{E}^2\,dv$$ and we have, $$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\mathbf{Q}}{\varepsilon\,\mathbf{A}}$$

Therefore, $$w_E = \frac{1}{2}\,\varepsilon\int_v \frac{\mathbf{Q}^2}{\varepsilon^2\,\mathbf{A}^2}\,dv = \frac{1}{2}\,\varepsilon\,\frac{\mathbf{Q}^2\,\mathbf{A}\,d}{\varepsilon^2\,\mathbf{A}^2} = \frac{1}{2}\,\frac{\mathbf{Q}^2\,d}{\varepsilon\,\mathbf{A}}$$ and $$\mathbf{C} = \frac{\varepsilon\,\mathbf{A}}{d}$$

Hence, $$\boxed{w_E = \frac{1}{2}\,\frac{\mathbf{Q}^2}{\mathbf{C}} = \frac{1}{2}\,\mathbf{Q}\,\mathbf{V} = \frac{1}{2}\,\mathbf{C}\,\mathbf{V}^2}$$

For the calculation of the resistance, we have a relationship between the capacitance and resistance, as given below:

$$\boxed{\mathbf{R}\,\mathbf{C} = \frac{\varepsilon}{\sigma} \quad \Rightarrow\Rightarrow \quad \mathbf{R} = \frac{\varepsilon}{\sigma\,\mathbf{C}} = \frac{\varepsilon}{\frac{\sigma\,\varepsilon\,\mathbf{A}}{d}}}$$

hence, $$\boxed{\mathbf{R} = \frac{d}{\sigma\,\mathbf{A}}}$$

Example (7): Find the capacitance and resistance between two concentric spherical conductors of conductivity (σ), and with the inner and outer radius of (a) and (b) respectively.

Solution:

Since the capacitance is given by: $\mathbf{C}=\dfrac{\mathbf{Q}}{\mathbf{V}}$ **and the charge consists with this defining region is calculated by using Gauss's law as:**

$$\mathbf{Q}=\varepsilon \oint_s \vec{\mathbf{E}}\cdot\vec{\mathbf{ds}}=\varepsilon \int_0^{2\pi}\int_0^{\pi} \mathbf{E}_r\,\hat{a}_r\cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r=\mathbf{E}_r\,(4\pi\varepsilon r^2)$$

hence, $$\vec{\mathbf{E}}=\frac{\mathbf{Q}}{4\pi\varepsilon r^2}\hat{a}_r\quad ---(1)$$

$$\mathbf{V}=-\int_b^a \vec{\mathbf{E}}\cdot\vec{\mathbf{dl}}=-\int_b^a \frac{\mathbf{Q}}{4\pi\varepsilon r^2}\hat{a}_r\cdot dr\,\hat{a}_r \Rightarrow\Rightarrow$$

hence, $$\mathbf{V}=\left(\frac{\mathbf{Q}}{4\pi\varepsilon r}\right)\Big|_b^a=\left(\frac{\mathbf{Q}}{4\pi\varepsilon}\right)\left[\frac{1}{a}-\frac{1}{b}\right]$$

Since, $\mathbf{C}=\dfrac{\mathbf{Q}}{\mathbf{V}}$ **, therefore, the capacitance of this configuration is:**

$$\mathbf{C}=\frac{\mathbf{Q}}{\mathbf{V}}=\frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}$$

and the resistance is:

$$\mathbf{R}\,\mathbf{C}=\frac{\varepsilon}{\sigma}=\frac{\varepsilon}{\sigma\mathbf{C}}\Rightarrow\Rightarrow \quad \mathbf{R}=\frac{\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\sigma}$$

This type of calculation is called (Q- method). But when finding (V) through Laplace's equation and then finding (Q) is called (V-method).

Example (8): Find the capacitance and resistance of a coaxial capacitor of length (L) and conductivity (σ), where the inner conductor has a radius of (a) and outer conductor has a radius of (b).

Solution:

We have: $\mathbf{C} = \dfrac{\mathbf{Q}}{\mathbf{V}} = \dfrac{\varepsilon \oint_s \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}}}{\int_L \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}}$ **then the value of (Q) is calculated through the use of**

Gauss's law as: $\mathbf{Q} = \varepsilon \oint_s \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{ds}} = \int_0^{2\pi} \int_0^{L} \mathbf{E}_\rho \hat{a}_\rho \cdot \rho \, d\phi \, dz \, \hat{a}_\rho \Rightarrow\Rightarrow$ $\mathbf{Q} = \mathbf{E}_\rho (2\pi \varepsilon \rho L)$ **and hence,**

$$\vec{\mathbf{E}} = \frac{\mathbf{Q}}{2\pi \varepsilon \rho L} \hat{a}_\rho \qquad \mathbf{V} = -\int_b^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\int_b^a \frac{\mathbf{Q}}{2\pi \varepsilon \rho L} \hat{a}_\rho \cdot d\rho \, \hat{a}_\rho \Rightarrow\Rightarrow \qquad \mathbf{V} = -\frac{\mathbf{Q}}{2\pi \varepsilon L} \left[\ln(\rho) \Big|_b^a \right]$$

$\mathbf{V} = \dfrac{\mathbf{Q}}{2\pi \varepsilon L} \ln\left(\dfrac{b}{a}\right)$ **Thus the capacitance of coaxial capacitor is:** $\mathbf{C} = \dfrac{\mathbf{Q}}{\mathbf{V}} = \dfrac{2\pi \varepsilon L}{\ln(b/a)}$

Therefore, the resistance can be calculated from the equation:

$$\mathbf{R}\,\mathbf{C} = \frac{\varepsilon}{\sigma} = \frac{\varepsilon}{\sigma \mathbf{C}} \Rightarrow\Rightarrow \qquad \mathbf{R} = \frac{\ln(b/a)}{2\pi \sigma L}$$

Example (9): A metal bar of conductivity (σ) is bent to form a flat (90°) sector of inner radius (a) and outer radius (b), and thickness (t), then show that:

(a). The resistance of the bar between the vertical curved surface at $\rho = a$ *and* $\rho = b$ **is given by:**

$$\mathbf{R} = \frac{2\ln\left(\frac{b}{a}\right)}{\sigma\,\pi\,t}$$

(b). The resistance between the two horizontal surfaces at $z = 0$ *and* $z = t$ **is given by:**

$$\mathbf{R} = \frac{4\,t}{\sigma\,\pi\,(b^2 - a^2)}$$

Solution:

(a). we have:

$$\mathbf{R} = \frac{-\int \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}}{\sigma \oint_s \vec{\mathbf{E}} \cdot \overrightarrow{d\mathbf{s}}} ------------(1)$$

$$\mathbf{I} = \sigma \int_0^{\pi/2} \int_0^{t} \mathbf{E}_\rho\, \hat{a}_\rho \cdot \rho\, d\phi\, dz\ \hat{a}_\rho = \mathbf{E}_\rho \left(\frac{\pi}{2}\rho\, t\, \sigma\right) \Rightarrow\Rightarrow$$ **hence,** $$\boxed{\vec{\mathbf{E}} = \frac{2\,\mathbf{I}}{\pi\,\rho\,t\,\sigma}\hat{a}_\rho}$$

$$\mathbf{V} = -\int_b^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\int_b^a \frac{2\mathbf{I}}{\pi\,\rho\,\sigma\,t}\,\hat{a}_\rho \cdot d\rho\ \hat{a}_\rho \Rightarrow\Rightarrow \mathbf{V} = -\frac{2\mathbf{I}}{\pi\,\sigma\,t}\left(\ln\rho\right)\Big|_b^a \Rightarrow\Rightarrow \boxed{\mathbf{V} = \frac{2\mathbf{I}}{\pi\,\sigma\,t}\ln\left(\frac{b}{a}\right)}$$

Hence, according to equation (1), the resistance between the two vertical surfaces of this metal bar is:

$$\mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{2\ln\left(\frac{b}{a}\right)}{\pi \sigma t}$$

(b). $\mathbf{I} = \sigma \int_0^{\pi/2} \int_b^a \mathbf{E}_z \, \hat{a}_z \cdot \rho \, d\rho \, d\phi \, \hat{a}_z \Rightarrow\Rightarrow \mathbf{E}_z \left(\frac{\pi}{2} \sigma\right) \left(\frac{\rho^2}{2}\right)\Big|_b^a \Rightarrow\Rightarrow \mathbf{I} = -\mathbf{E}_z \frac{\pi}{4} \sigma \left(b^2 - a^2\right)$

Hence, $\vec{\mathbf{E}} = \frac{4\mathbf{I}}{\pi \sigma (b^2 - a^2)} \hat{a}_z$ $\quad \mathbf{V} = -\int_b^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\int_0^t -\left(\frac{4\mathbf{I}}{\pi \sigma (b^2 - a^2)} \hat{a}_z\right) \cdot dz \, \hat{a}_z \Rightarrow\Rightarrow$ $\mathbf{V} = \frac{4\mathbf{I} \, t}{\pi \sigma (b^2 - a^2)}$

And according to equation (1), the resistance between the two horizontal surface of this metal bar is:

$$\mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{4t}{\pi \sigma (b^2 - a^2)}$$

Example (10): A coaxial cable contains an insulating material of ($\sigma 1$) in its upper half and another material of conductivity ($\sigma 2$) in its lower half. If the radius of the center wire is (a) and that of the sheath is (b), then show that the leakage resistance of length (L) of the cable is:

$$\mathbf{R} = \frac{\ln\left(\frac{b}{a}\right)}{\pi L(\sigma_1 + \sigma_2)}$$

Solution:

We have: $$\mathbf{R} = \frac{-\int \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}}}{\sigma \oint_s \vec{\mathbf{E}} \cdot \overrightarrow{d\mathbf{s}}} ----------- (1)$$

$$\mathbf{I} = \sigma_1 \int_0^{\pi} \int_0^{L} \mathbf{E}_\rho \, \hat{a}_\rho \cdot \rho \, d\phi \, dz \; \hat{a}_\rho + \sigma_2 \int_{\pi}^{2\pi} \int_0^{L} \mathbf{E}_\rho \, \hat{a}_\rho \cdot \rho \, d\phi \, dz \; \hat{a}_\rho$$

$$\mathbf{I} = \mathbf{E}_\rho (\sigma_1 \rho \pi L) + \mathbf{E}_\rho (\sigma_2 \rho \pi L) = \mathbf{E}_\rho \rho \pi L(\sigma_1 + \sigma_2) \Rightarrow\Rightarrow$$

hence, $$\vec{\mathbf{E}} = \frac{\mathbf{I}}{\rho \pi L(\sigma_1 + \sigma_2)} \hat{a}_\rho$$

$$\mathbf{V} = -\int_b^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\int_b^a \frac{\mathbf{I}}{\rho \pi L(\sigma_1 + \sigma_2)} \hat{a}_\rho \cdot d\rho \; \hat{a}_\rho \Rightarrow\Rightarrow$$

hence, $$\mathbf{V} = -\frac{\mathbf{I}}{\pi L(\sigma_1 + \sigma_2)} \ln(\rho)\Big|_b^a$$

$$\mathbf{V} = \frac{\mathbf{I}}{\pi L(\sigma_1 + \sigma_2)} \ln(b/a)$$

According to eq.(1) the resistance is given as: $$\mathbf{R} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\ln(b/a)}{\pi L(\sigma_1 + \sigma_2)}$$

And the capacitance is evaluated by the equation: $$\mathbf{R}\,\mathbf{C} = \frac{\varepsilon}{\sigma}$$ but $$\varepsilon = \varepsilon_\circ\ \varepsilon_r$$

Therefore, $$\varepsilon = \varepsilon_\circ (\varepsilon_{r1} + \varepsilon_{r2})$$ and hence,

$$\mathbf{C} = \frac{\varepsilon}{\mathbf{R}\sigma} = \frac{\pi L \varepsilon_\circ (\varepsilon_{r1} + \varepsilon_{r2})\,(\sigma_1 + \sigma_2)}{(\sigma_1 + \sigma_2)\,\ln(b/a)}$$

$$\mathbf{C} = \frac{\varepsilon}{\mathbf{R}\sigma} = \frac{\pi L \varepsilon_\circ (\varepsilon_{r1} + \varepsilon_{r2})}{\ln(b/a)}$$

Home Work

Q_1/ If the ends of a cylindrical bar of Carbon with $\sigma = 3\times10^4\ S/m$ **of radius** $r = 5mm \quad and \quad length \quad l = 8\ cm$ **are maintained at a potential difference of** $V = 9\ volt$ **, find :**

a. The resistance of the bar **b.** The current through the bar **c.** The power dissipated in the bar

Q_2/ For a coaxial cable of inner conductor radius (a) and outer conductor radius (b) and a dielectric ε_r **in between, assume a charge density** $\rho_v = \frac{\rho}{\rho_\circ}$ **is added in the dielectric region. Use Poisson equation to derive an expression for V and E-field and calculate surface charge density on each plate?**

Q_3/ A conical section of material extends over the range $2\ cm \le r \le 9\ cm \quad for \quad 0 \le \theta \le 30^\circ \quad and \quad \sigma = 0.02\ (S/m)$ **if conductive plates are added at each radial end of the section. Determine the resistance and capacitance of the section?**

Q_4/ A spherical capacitance has inner radius (a) and outer radius (b) and filled with an inhomogeneous dielectric with $\varepsilon = \varepsilon_\circ \frac{k}{r^2}$ **, show that the capacitance of the capacitor is given by:**

$$C = \frac{4\pi\varepsilon_\circ k}{b-a}$$

Q_5/ If the earth is regarded as a spherical capacitor, what is its capacitance? Assume the radius of the earth to be approximately $r = 6370\ Km$? Ans. C = 0.7078 mF

Q_6/ Find the capacitance and resistance of a coaxial capacitor of length (L) and conductivity (σ), where the inner conductor has radius (a) and the outer has a radius (b) ??

Q_7/ Find the capacitance and resistance between two concentric spherical conductors of conductivity (σ), where the sphere has an inner radius of (a) and outer radius of (b) ??

Q_8/ A coaxial cable contains an insulating material of conductivity σ_1 in it's upper half and another material of conductivity σ_2 in it's lower half. If the radius of the center wire is (a) and that of the sheath is (b), show that the leakage resistance of length (L) of the cable is:

$$R = \frac{\ln\left(\frac{b}{a}\right)}{L\pi(\sigma_1 + \sigma_2)}$$

4-10: Boundary Conditions:

So far we have considered the existence of the electric field in a homogeneous medium. Now, if the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions. The conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. We shall consider the boundary conditions at an interface separating,

- **dielectric (ε_{r1}) and dielectric (ε_{r2})**
- **conductor and dielectric**
- **conductor and free space**

To determine the boundary conditions, we need to use Maxwell's equation for electric and magnetic fields: For static electric field we have two equation of Maxwell, they are:

$$\oint_l \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 \quad \Rightarrow\Rightarrow \quad \vec{\nabla} \times \vec{\mathbf{E}} = 0$$

and

$$\oint_s \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{d}s} = Q_{enc} \quad \Rightarrow\Rightarrow \quad \vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_v$$

Also we need to decompose the electric field intensity into two orthogonal components; they are as follows:

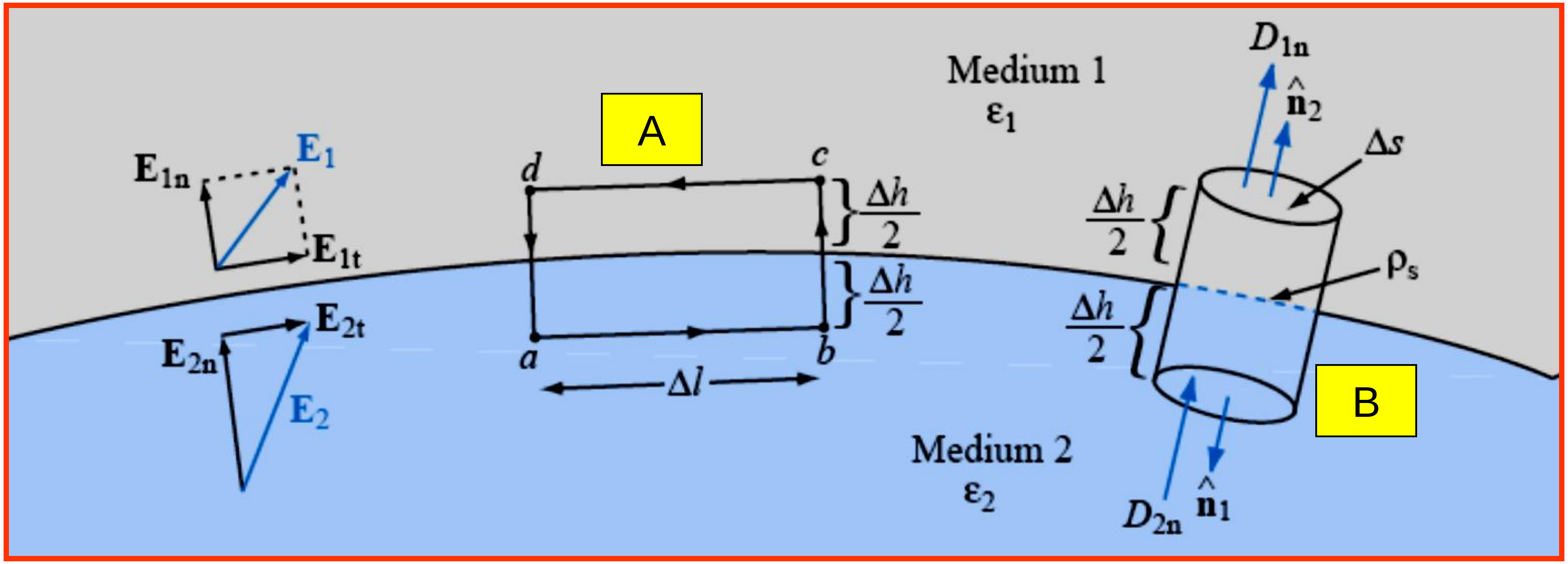
$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{\tan gential} + \vec{\mathbf{E}}_{normal} = \vec{\mathbf{E}}_t + \vec{\mathbf{E}}_n$$

and,

$$\vec{\mathbf{D}} = \vec{\mathbf{D}}_{\tan gential} + \vec{\mathbf{D}}_{normal} = \vec{\mathbf{D}}_t + \vec{\mathbf{D}}_n$$

4-10-1: Dielectric-Dielectric Boundary Conditions:

Consider the ($\vec{\mathbf{E}} - field$) exist in a region consist of two different dielectrics, characterized by: ($\varepsilon_1 = \varepsilon_\circ \varepsilon_{r1}$ *and* $\varepsilon_2 = \varepsilon_\circ \varepsilon_{r2}$), then:



$$\vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_{1t} + \vec{\mathbf{E}}_{1n} \quad ------(1)$$

and

$$\vec{\mathbf{E}}_2 = \vec{\mathbf{E}}_{2t} + \vec{\mathbf{E}}_{2n} \quad ------(2)$$

From the figure (A) closed path (abcda) we can infer that:

$$\oint_C \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = \int_a^b \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} + \int_b^c \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} + \int_d^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 \quad ------(3)$$

At ($h \to 0$), at the interface between two media, then:

$$\int_b^c \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = \int_d^a \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 \quad --------(4)$$

thus,

$$\int_a^b \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 ------(5)$$

Hence, from eq.(5) we can obtain: $\vec{\mathbf{E}}_{1t}\, l - \vec{\mathbf{E}}_{2t}\, l = 0 \quad \Rightarrow\Rightarrow$

$$\vec{\mathbf{E}}_{1t} = \vec{\mathbf{E}}_{2t} --------(6)$$

Since, $\vec{\mathbf{D}} = \varepsilon_\circ\ \varepsilon_r\ \vec{\mathbf{E}}$ therefore,

$$\frac{\vec{\mathbf{D}}_{1t}}{\varepsilon_{r1}} = \frac{\vec{\mathbf{D}}_{2t}}{\varepsilon_{r2}} \quad -------(7)$$

From figure (B), we have:

$$\oint_s \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} = \oint_{top} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} + \oint_{side} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} + \oint_{bottom} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} = Q_{enc} \quad -----------(8)$$

At ($h \to 0$), at the interface between two media, then:

$$\oint_{side} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} = 0 ------------(9)$$

$$Q_{enc} = \oint_{top} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} + \oint_{bottom} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} = \vec{\mathbf{D}}_{1n}\,\Delta s - \vec{\mathbf{D}}_{2n}\,\Delta s$$

and also we have,

$$Q_{enc} = \rho_s\,\Delta s$$

Therefore, $(\vec{\mathbf{D}}_{1n} - \vec{\mathbf{D}}_{2n})\,\Delta s = \rho_s\;\Delta s \quad \Rightarrow\Rightarrow$ **hence,** $\vec{\mathbf{D}}_{1n} - \vec{\mathbf{D}}_{2n} = \rho_s \text{------}(10)$

If ($\rho_s = 0$ **) at the interface, then:** $\vec{\mathbf{D}}_{1n} = \vec{\mathbf{D}}_{2n} \text{------}(11)$ **and we have** $\vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}}$

Therefore, $\vec{\mathbf{D}}_{1n} = \varepsilon_{r1}\;\varepsilon_{\circ}\,\vec{\mathbf{E}}_{1n} \qquad \textit{and} \qquad \vec{\mathbf{D}}_{2n} = \varepsilon_{r2}\;\varepsilon_{\circ}\,\vec{\mathbf{E}}_{2n}$

, and obtaining the following relations:

$$\frac{\mathbf{E}_{1n}}{\varepsilon_{r1}} = \frac{\mathbf{E}_{2n}}{\varepsilon_{r2}} \text{-------------}(12)$$

According to the figure shown below and eq.(6) we can evaluate the angle of incidence and refraction between these two interfaces as given below:

$$\vec{\mathbf{E}}_{1t} = \vec{\mathbf{E}}_{2t} \Rightarrow\Rightarrow$$

hence, $$\mathbf{E}_1 \sin\theta_1 = \mathbf{E}_2 \sin\theta_2 ---------(13)$$

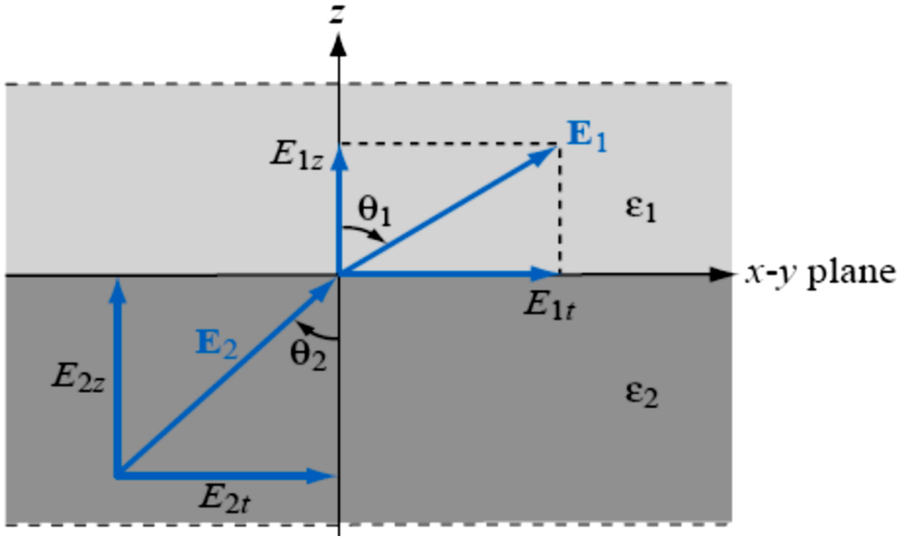
Also from eq.(12), we can write:

$$\varepsilon_{r1}\ \mathbf{E}_{1n} = \varepsilon_{r2}\ \mathbf{E}_{2n}$$

hence, $$\varepsilon_{r1}\vec{\mathbf{E}}_1\ \cos\theta_1 = \varepsilon_{r2}\vec{\mathbf{E}}_2\ \cos\theta_2 ----(14)$$

Dividing eq.(13) by eq.(14) we get:

$$\frac{\sin\theta_1}{\varepsilon_{r1}\ \cos\ \theta_1} = \frac{\sin\theta_2}{\varepsilon_{r2}\ \cos\ \theta_2} \Rightarrow\Rightarrow$$



hence,

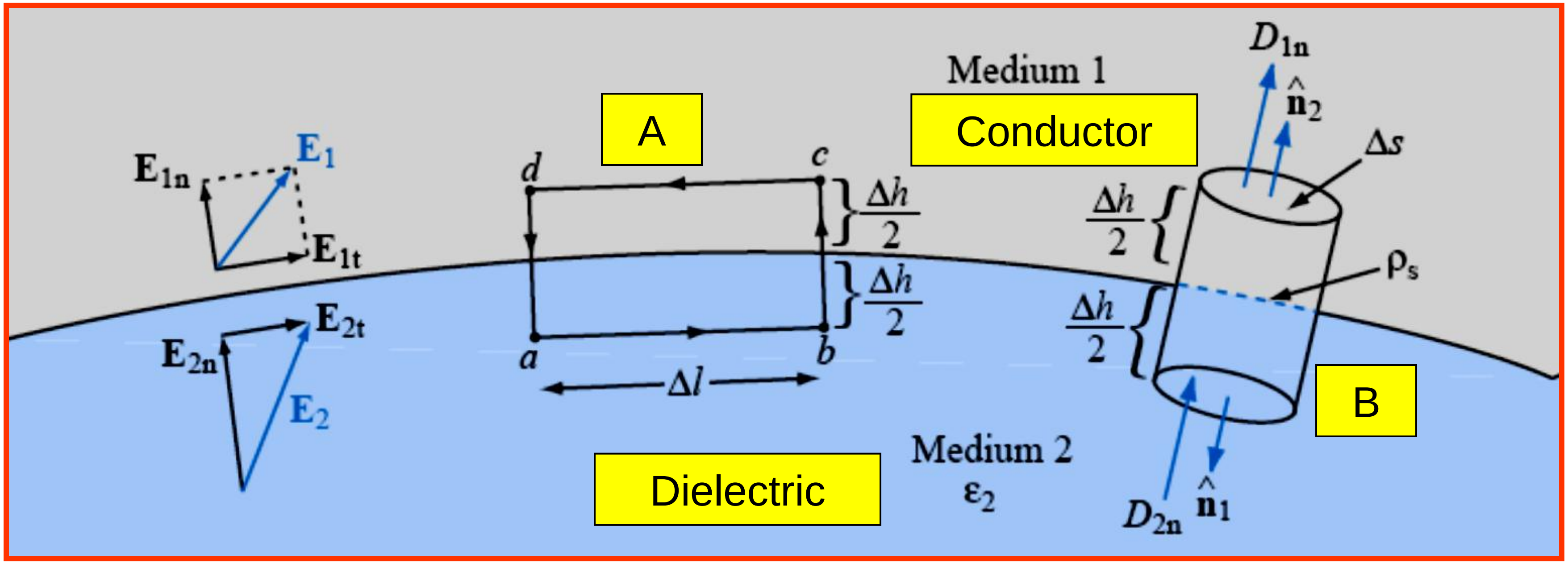
$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} ----------(15)$$

Eq.(15) is the law of refraction of the electric field at the boundary free of charge ($\rho_s = 0$). Thus, an interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the sides of the interface.

4-10-2: Conductor-Dielectric Boundary Conditions:

The conductor is assumed to be perfect (i.e. $\sigma \to \infty$). Although, such a conductor is not practically realizable, we may regard conductors such as Copper and Silver as though they were perfect conductors.

To determine the boundary conditions for a conductor-dielectric interface, we follow the same procedure used for dielectric-dielectric interface except that we incorporate the fact that inside the conductor $\vec{\mathbf{E}} = 0$



Applying: $$\oint_C \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = \int_a^b \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_b^c \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0$$

At ($h \to 0$), at the interface between two media, then:

$$\int_b^c \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = \int_d^a \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0$$

and,

$$\int_a^b \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} + \int_c^d \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0$$

But inside the conductor $\vec{\mathbf{E}} = 0$, thus: $\int_c^d \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0$

Therefore, $\int_a^b \vec{\mathbf{E}}\cdot\overrightarrow{\mathbf{dl}} = 0$ And hence, $\vec{\mathbf{E}}_t = 0 \quad and \quad hence, \qquad \vec{\mathbf{D}}_t = 0 ------(2)$

Applying: $\oint_s \vec{\mathbf{D}}\cdot\overrightarrow{\mathbf{ds}} = \oint_{top} \vec{\mathbf{D}}\cdot\overrightarrow{\mathbf{ds}} + \oint_{side} \vec{\mathbf{D}}\cdot\overrightarrow{\mathbf{ds}} + \oint_{bottom} \vec{\mathbf{D}}\cdot\overrightarrow{\mathbf{ds}} = Q_{enc}$ we get: $\oint_s \vec{\mathbf{D}}\cdot\overrightarrow{\mathbf{ds}} = \vec{\mathbf{D}}_n\,\Delta s + 0 + 0\times\Delta s = \rho_s\,\Delta s$

$$\Rightarrow\Rightarrow \quad \vec{\mathbf{D}}_n = \rho_s \quad and \quad hence, \quad \vec{\mathbf{E}}_n = \frac{\rho_s}{\varepsilon_\circ\,\varepsilon_r} ---------(3)$$

Thus, the boundary conditions for conductor-dielectric boundary conditions are:

$$\vec{\mathbf{E}}_t = \vec{\mathbf{D}}_t = 0 \qquad and \qquad \vec{\mathbf{E}}_n = \frac{\vec{\mathbf{D}}_n}{\varepsilon_\circ\,\varepsilon_r} = \frac{\rho_s}{\varepsilon_\circ\,\varepsilon_r}$$

Thus, under static conditions, the following conclusions can be made about a perfect conductor:

(1). **No electric field may exist within a conductor; that is ($\rho_v = 0 \quad and \quad \vec{\mathbf{E}} = 0$).**

(2). **Since ($\vec{\mathbf{E}} = -\vec{\nabla}V = 0$), so there can be no potential difference between any two points in the conductor, that is; conductor is an equipotential body.**

(3). **The electric field intensity can be external to the conductor and normal to its surface; that is:**

$$\vec{\mathbf{D}}_t = \varepsilon_\circ \varepsilon_r \vec{\mathbf{E}}_t = 0 \qquad and \qquad \vec{\mathbf{D}}_n = \varepsilon_\circ \varepsilon_r \vec{\mathbf{E}}_n = \rho_s$$

(4). **An important application of the fact that ($\vec{\mathbf{E}} = 0$) inside a conductor is in electrostatic screening or shielding.**

4-10-3: Conductor-Free space Boundary Conditions:

This is a special case of the conductor-dielectric boundary conditions, by replacing ($\varepsilon_{r1} = 1$) because a free space is a special dielectric medium for which ($\varepsilon_{r1} = 1$). Thus, the boundary conditions are:

$$\vec{\mathbf{D}}_t = \varepsilon_\circ \vec{\mathbf{E}}_t = 0 \qquad and \qquad \vec{\mathbf{D}}_n = \varepsilon_\circ \vec{\mathbf{E}}_n = \rho_s$$

It should be noted that these equations implies that ($\vec{\mathbf{E}} - field$) must approach a conducting surface normally.

Example(11): Two extensive homogenous isotropic dielectrics meet on plane ($z = 0$, $for \quad z \geq 0 \quad \varepsilon_{r1} = 4$ $and \quad for \quad z \leq 0$, $\varepsilon_{r2} = 3$). A uniform electric field ($\vec{\mathbf{E}} = 5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z$) Kv/m exist for ($z \geq 0$), then find:

(a). $\vec{\mathbf{E}}_2 \quad for \quad z \leq 0$

(b). The angles ($\vec{\mathbf{E}}_1 \quad and \quad \vec{\mathbf{E}}_2$ **) makes with the interface**

(c). The energy densities in (J/m3) in both medium.

(d). The energy within a cube of side (2 m), centered at point (3,4,-5).

Solution: **(a).** $\vec{\mathbf{E}}_2 \; for \;\; z \le 0$ **?**

$$\vec{\mathbf{E}}_2 = \vec{\mathbf{E}}_{2t} + \vec{\mathbf{E}}_{2n} \quad and \quad \vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_{1t} + \vec{\mathbf{E}}_{1n}$$

$$\vec{\mathbf{E}}_{1t} = \vec{\mathbf{E}}_{2t} = 5\hat{a}_x - 2\hat{a}_y \quad and \quad \vec{\mathbf{E}}_{1n} = 3\,\hat{a}_z$$

According to the boundary conditions:

$$\vec{\mathbf{E}}_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\vec{\mathbf{E}}_{1n} \Rightarrow\Rightarrow \vec{\mathbf{E}}_{2n} = \frac{4}{3}\times 3\hat{a}_z \Rightarrow\Rightarrow$$

$$\vec{\mathbf{E}}_{2n} = 4\,\hat{a}_z \quad \textbf{and} \quad \vec{\mathbf{E}}_2 = \vec{\mathbf{E}}_{2t} + \vec{\mathbf{E}}_{2n}$$

Hence, $\vec{\mathbf{E}}_2 = 5\hat{a}_x - 2\,\hat{a}_y + 4\,\hat{a}_z$

(b).

$$\left|\vec{\mathbf{E}}_{2t}\right| = \sqrt{25+4} = \sqrt{29} \quad and \quad \left|\vec{\mathbf{E}}_{2n}\right| = \sqrt{16} = 4$$

$$\left|\vec{\mathbf{E}}_{1t}\right| = \sqrt{25+4} = \sqrt{29} \quad and \quad \left|\vec{\mathbf{E}}_{1n}\right| = \sqrt{9} = 3$$

$$\alpha_1 = 90^\circ - \theta_1 \qquad and \qquad \alpha_2 = 90^\circ - \theta_2$$

$$\tan\theta_1 = \frac{\left|\vec{\mathbf{E}}_{1t}\right|}{\left|\vec{\mathbf{E}}_{1n}\right|} = \frac{\sqrt{29}}{3} \Rightarrow\Rightarrow \theta_1 = 60.9^\circ \Rightarrow\Rightarrow \alpha_1 = 29.1^\circ$$

$$\tan\theta_2 = \frac{\left|\vec{\mathbf{E}}_{2t}\right|}{\left|\vec{\mathbf{E}}_{2n}\right|} = \frac{\sqrt{29}}{4} \Rightarrow\Rightarrow \theta_2 = 53.4^\circ \Rightarrow\Rightarrow \alpha_1 = 36.6^\circ$$

(c). The energy densities are given by:

$$w_1 = \frac{1}{2}\varepsilon_\circ\,\varepsilon_{r1}\left|\vec{\mathbf{E}}_1\right|^2 = \frac{1}{2}\times\frac{10^{-9}}{36\pi}\times 3\times(25+4+9) = 672\ J/m^3$$

$$w_2 = \frac{1}{2}\varepsilon_\circ\,\varepsilon_{r2}\left|\vec{\mathbf{E}}_2\right|^2 = \frac{1}{2}\times\frac{10^{-9}}{36\pi}\times 4\times(25+4+16) = 597\ J/m^3$$

(d). At the center (3,4,-5) of the cube of side (2 m) ($z = -5 \subset 0$), that is the cube is in the region (2) and: ($2 \le x \le 4 \quad , \quad 3 \le y \le 5 \quad and \quad -6 \le z \le -4$), hence:

$$w_E = \frac{1}{2}\varepsilon_\circ\ \varepsilon_{r2}\int_2^4\int_3^3\int_{-6}^{-4}\left|\vec{\mathbf{E}}_2\right|^2 dx\,dy\,dz \Rightarrow\Rightarrow \quad w_E = 4.776\ mJ$$

Home work

Q$_1$/ Two extensive homogeneous isotropic dielectrics meet on plane $z=0$. For $z \geq 0,\ \varepsilon_{r1}=4$ *and* *for* $z \leq 0,\ \varepsilon_{r2}=3$. A uniform electric field $\vec{E}=(5\hat{a}_x-2\hat{a}_y+3\hat{a}_z)\dfrac{kV}{m}$ exists for $z \geq 0$, then find:

a. $\vec{E}_2$ *for* $z \leq 0$

b. The angles $\vec{E}_1$ *and* $\vec{E}_2$ makes with the interface ?

c. The energy densities in (J/m3) in both medium?

d. The energy within a cube of side (2 m) centered at (3,4,-5)?

Q$_2$/ For $y<0\quad \varepsilon_{r1}=4$ *and* $\vec{E}_1=3\hat{a}_x+6\pi\,\hat{a}_y+4\,\hat{a}_z\ (V/m)$, at $y=0\quad \rho_s=0.25\ (nC/m^2)$, if $\varepsilon_{r2}=5$ *for* $y>0$, then find? $\vec{\mathbf{E}}_2, \vec{\mathbf{D}}_2, \vec{\mathbf{D}}_1, \vec{\mathbf{P}}_1, \vec{\mathbf{P}}_2$ *and* the angle which makes $\vec{E}_1$ *and* $\vec{E}_2$ with the interface

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Q$_3$/ Region $z<0$ contain a dielectric for which $\varepsilon_{r1}=2.5$ while region $z>0$ is characterized by $\varepsilon_{r1}=4$. Let $\vec{E}_1=-30\,\hat{a}_x+50\hat{a}_y+70\,\hat{a}_z\ (V/m)$, then find : $\vec{E}_2$, $\vec{D}_1$, $\vec{D}_2$, $\vec{P}_1$, $\vec{P}_2$ and the energy density in both region?