Chapter 2

COORDINATE SYSTEMS AND TRANSFORMATION

Education makes a people easy to lead, but difficult to drive; easy to govern but impossible to enslave.

—HENRY P. BROUGHAM

2.1 INTRODUCTION

In general, the physical quantities we shall be dealing with in EM are functions of space and time. In order to describe the spatial variations of the quantities, we must be able to define all points uniquely in space in a suitable manner. This requires using an appropriate coordinate system.

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or nonorthogonal.

An **orthogonal system** is one in which the coordinates are mutually perpendicular.

Nonorthogonal systems are hard to work with and they are of little or no practical use. Examples of orthogonal coordinate systems include the Cartesian (or rectangular), the circular cylindrical, the spherical, the elliptic cylindrical, the parabolic cylindrical, the conical, the prolate spheroidal, the oblate spheroidal, and the ellipsoidal.[1] A considerable amount of work and time may be saved by choosing a coordinate system that best fits a given problem. A hard problem in one coordi nate system may turn out to be easy in another system.

In this text, we shall restrict ourselves to the three best-known coordinate systems: the Cartesian, the circular cylindrical, and the spherical. Although we have considered the Cartesian system in Chapter 1, we shall consider it in detail in this chapter. We should bear in mind that the concepts covered in Chapter 1 and demonstrated in Cartesian coordinates are equally applicable to other systems of coordinates. For example, the procedure for

[1]For an introductory treatment of these coordinate systems, see M. R. Spigel, *Mathematical Handbook of Formulas and Tables.* New York: McGraw-Hill, 1968, pp. 124–130.

finding dot or cross product of two vectors in a cylindrical system is the same as that used in the Cartesian system in Chapter 1.

Sometimes, it is necessary to transform points and vectors from one coordinate system to another. The techniques for doing this will be presented and illustrated with examples.

2.2 CARTESIAN COORDINATES (X, Y, Z)

As mentioned in Chapter 1, a point P can be represented as (x, y, z) as illustrated in Figure 1.1. The ranges of the coordinate variables x, y, and z are

$$\begin{aligned} -\infty &< x < \infty \\ -\infty &< y < \infty \\ -\infty &< z < \infty \end{aligned} \tag{2.1}$$

A vector $\mathbf{A}$ in Cartesian (otherwise known as rectangular) coordinates can be written as

$$(A_x, A_y, A_z) \qquad \text{or} \qquad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \tag{2.2}$$

where $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are unit vectors along the x-, y-, and z-directions as shown in Figure 1.1.

2.3 CIRCULAR CYLINDRICAL COORDINATES (ρ, ϕ, z)

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Figure 2.1. Observe Figure 2.1 closely and note how we define each space variable: ρ is the radius of the cylinder passing through P or the radial distance from the z-axis: ϕ, called the

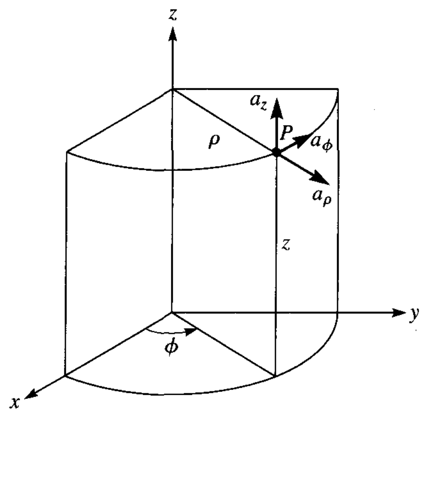

Figure 2.1 Point P and unit vectors in the cylindrical coordinate system.

azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$$\begin{aligned} 0 &\leq \rho < \infty \\ 0 &\leq \phi < 2\pi \\ -\infty &< z < \infty \end{aligned} \tag{2.3}$$

A vector $\mathbf{A}$ in cylindrical coordinates can be written as

$$(A_\rho, A_\phi, A_z) \qquad \text{or} \qquad A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z \tag{2.4}$$

where $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are unit vectors in the ρ-, ϕ-, and z-directions as illustrated in Figure 2.1. Note that $\mathbf{a}_\phi$ is not in degrees; it assumes the unit vector of $\mathbf{A}$. For example, if a force of 10 N acts on a particle in a circular motion, the force may be represented as $\mathbf{F} = 10\mathbf{a}_\phi$ N. In this case, $\mathbf{a}_\phi$ is in newtons.

The magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2} \tag{2.5}$$

Notice that the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are mutually perpendicular because our coordinate system is orthogonal; $\mathbf{a}_\rho$ points in the direction of increasing ρ, $\mathbf{a}_\phi$ in the direction of increasing ϕ, and $\mathbf{a}_z$ in the positive z-direction. Thus,

$$\mathbf{a}_\rho \cdot \mathbf{a}_\rho = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \tag{2.6a}$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\rho = 0 \tag{2.6b}$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z \tag{2.6c}$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho \tag{2.6d}$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi \tag{2.6e}$$

where eqs. (2.6c) to (2.6e) are obtained in cyclic permutation (see Figure 1.9).

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from Figure 2.2 as

$$\boxed{\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z} \tag{2.7}$$

or

$$\boxed{x = \rho \cos \phi, \qquad y = \rho \sin \phi, \qquad z = z} \tag{2.8}$$

Whereas eq. (2.7) is for transforming a point from Cartesian (x, y, z) to cylindrical (ρ, ϕ, z) coordinates, eq. (2.8) is for $(\rho, \phi, z) \rightarrow (x, y, z)$ transformation.

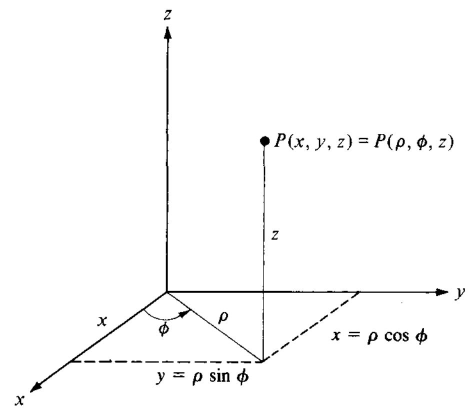

Figure 2.2 Relationship between (x, y, z) and (ρ, ϕ, z).

The relationships between $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ and $(\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z)$ are obtained geometrically from Figure 2.3:

$$\begin{aligned}
\mathbf{a}_x &= \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi \\
\mathbf{a}_y &= \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi \\
\mathbf{a}_z &= \mathbf{a}_z
\end{aligned} \tag{2.9}$$

or

$$\begin{aligned}
\mathbf{a}_\rho &= \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y \\
\mathbf{a}_\phi &= -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y \\
\mathbf{a}_z &= \mathbf{a}_z
\end{aligned} \tag{2.10}$$

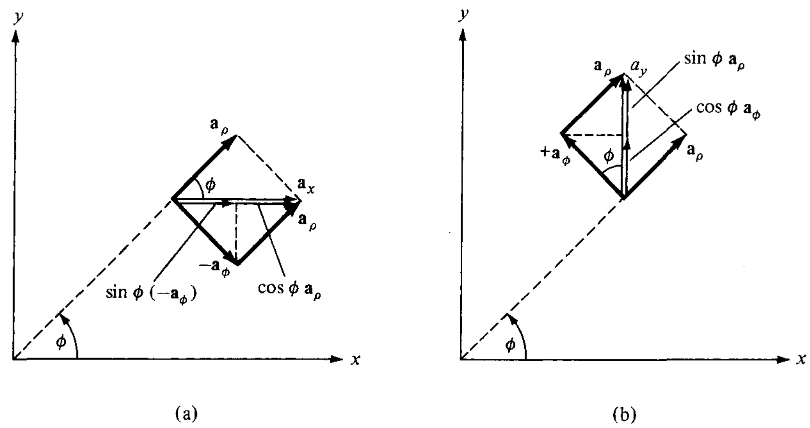

Figure 2.3 Unit vector transformation: **(a)** cylindrical components of $\mathbf{a}_x$, **(b)** cylindrical components of $\mathbf{a}_y$.

Finally, the relationships between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are obtained by simply substituting eq. (2.9) into eq. (2.2) and collecting terms. Thus

$$\mathbf{A} = (A_x \cos\phi + A_y \sin\phi)\mathbf{a}_\rho + (-A_x \sin\phi + A_y \cos\phi)\mathbf{a}_\phi + A_z\mathbf{a}_z \tag{2.11}$$

or

$$\begin{aligned} A_\rho &= A_x \cos\phi + A_y \sin\phi \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \\ A_z &= A_z \end{aligned} \tag{2.12}$$

In matrix form, we have the transformation of vector $\mathbf{A}$ from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as

$$\boxed{\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}} \tag{2.13}$$

The inverse of the transformation $(A_\rho, A_\phi, A_z) \to (A_x, A_y, A_z)$ is obtained as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \tag{2.14}$$

or directly from eqs. (2.4) and (2.10). Thus

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}} \tag{2.15}$$

An alternative way of obtaining eq. (2.14) or (2.15) is using the dot product. For example:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \mathbf{a}_x \cdot \mathbf{a}_\rho & \mathbf{a}_x \cdot \mathbf{a}_\phi & \mathbf{a}_x \cdot \mathbf{a}_z \\ \mathbf{a}_y \cdot \mathbf{a}_\rho & \mathbf{a}_y \cdot \mathbf{a}_\phi & \mathbf{a}_y \cdot \mathbf{a}_z \\ \mathbf{a}_z \cdot \mathbf{a}_\rho & \mathbf{a}_z \cdot \mathbf{a}_\phi & \mathbf{a}_z \cdot \mathbf{a}_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \tag{2.16}$$

The derivation of this is left as an exercise.

2.4 SPHERICAL COORDINATES (r, θ, ϕ)

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry. A point P can be represented as (r, θ, ϕ) and is illustrated in Figure 2.4. From Figure 2.4, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the *colatitude*) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$$\begin{aligned} 0 &\leq r < \infty \\ 0 &\leq \theta \leq \pi \\ 0 &\leq \phi < 2\pi \end{aligned} \tag{2.17}$$

A vector $\mathbf{A}$ in spherical coordinates may be written as

$$(A_r, A_\theta, A_\phi) \qquad \text{or} \qquad A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi \tag{2.18}$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are unit vectors along the r-, θ-, and ϕ-directions. The magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = (A_r^2 + A_\theta^2 + A_\theta^2)^{1/2} \tag{2.19}$$

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are mutually orthogonal; $\mathbf{a}_r$ being directed along the radius or in the direction of increasing r, $\mathbf{a}_\theta$ in the direction of increasing θ, and $\mathbf{a}_\phi$ in the direction of increasing ϕ. Thus,

$$\begin{aligned} \mathbf{a}_r \cdot \mathbf{a}_r &= \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = 1 \\ \mathbf{a}_r \cdot \mathbf{a}_\theta &= \mathbf{a}_\theta \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_r = 0 \\ \mathbf{a}_r \times \mathbf{a}_\theta &= \mathbf{a}_\phi \\ \mathbf{a}_\theta \times \mathbf{a}_\phi &= \mathbf{a}_r \\ \mathbf{a}_\phi \times \mathbf{a}_r &= \mathbf{a}_\theta \end{aligned} \tag{2.20}$$

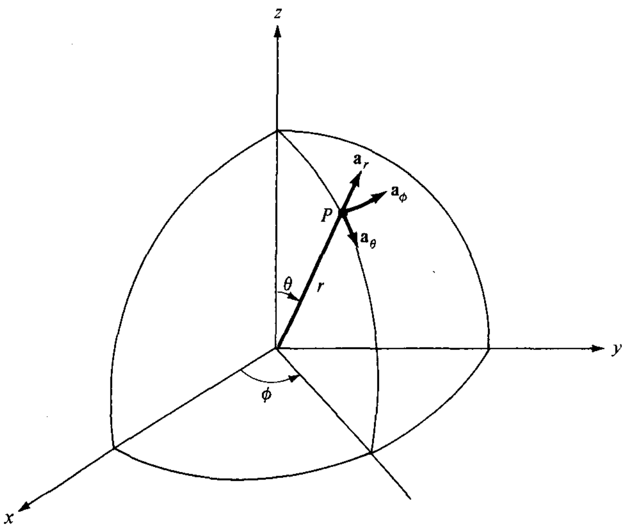

Figure 2.4 Point P and unit vectors in spherical coordinates.

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From Figure 2.5 it is easy to notice that

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}} \tag{2.21}$$

or

$$\boxed{x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta} \tag{2.22}$$

In eq. (2.21), we have $(x, y, z) \rightarrow (r, \theta, \phi)$ point transformation and in eq. (2.22), it is $(r, \theta, \phi) \rightarrow (x, y, z)$ point transformation.

The unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ and $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\phi$ are related as follows:

$$\begin{aligned}
\mathbf{a}_x &= \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi \\
\mathbf{a}_y &= \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi \\
\mathbf{a}_z &= \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta
\end{aligned} \tag{2.23}$$

or

$$\begin{aligned}
\mathbf{a}_r &= \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z \\
\mathbf{a}_\theta &= \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z \\
\mathbf{a}_\phi &= -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y
\end{aligned} \tag{2.24}$$

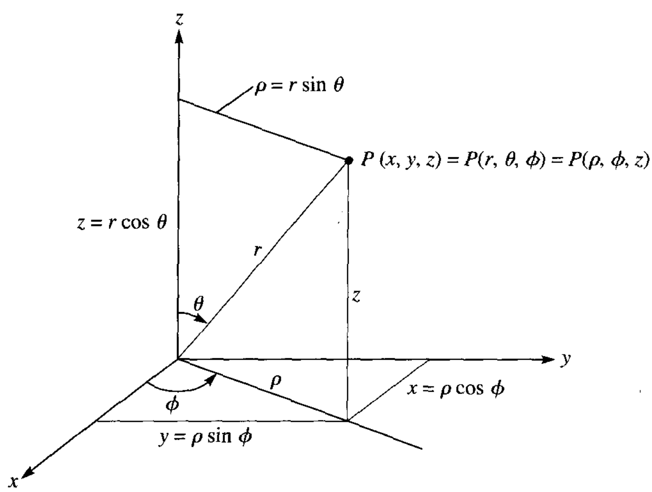

Figure 2.5 Relationships between space variables (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).

The components of vector $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{A} = (A_r, A_\theta, A_\phi)$ are related by substituting eq. (2.23) into eq. (2.2) and collecting terms. Thus,

$$\begin{aligned}\mathbf{A} = {} & (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta)\mathbf{a}_r + (A_x\cos\theta\cos\phi \\ & + A_y\cos\theta\sin\phi - A_z\sin\theta)\mathbf{a}_\theta + (-A_x\sin\phi + A_y\cos\phi)\mathbf{a}_\phi\end{aligned} \tag{2.25}$$

and from this, we obtain

$$\begin{aligned} A_r &= A_x \sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned} \tag{2.26}$$

In matrix form, the $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\boxed{\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}} \tag{2.27}$$

The inverse transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ is similarly obtained, or we obtain it from eq. (2.23). Thus,

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}} \tag{2.28}$$

Alternatively, we may obtain eqs. (2.27) and (2.28) using the dot product. For example,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \mathbf{a}_r\cdot\mathbf{a}_x & \mathbf{a}_r\cdot\mathbf{a}_y & \mathbf{a}_r\cdot\mathbf{a}_z \\ \mathbf{a}_\theta\cdot\mathbf{a}_x & \mathbf{a}_\theta\cdot\mathbf{a}_y & \mathbf{a}_\theta\cdot\mathbf{a}_z \\ \mathbf{a}_\phi\cdot\mathbf{a}_x & \mathbf{a}_\phi\cdot\mathbf{a}_y & \mathbf{a}_\phi\cdot\mathbf{a}_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{2.29}$$

For the sake of completeness, it may be instructive to obtain the point or vector transformation relationships between cylindrical and spherical coordinates using Figures 2.5 and 2.6 (where ϕ is held constant since it is common to both systems). This will be left as an exercise (see Problem 2.9). Note that in point or vector transformation the point or vector has not changed; it is only expressed differently. Thus, for example, the magnitude of a vector will remain the same after the transformation and this may serve as a way of checking the result of the transformation.

The distance between two points is usually necessary in EM theory. The distance d between two points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is generally given by

$$d = |\mathbf{r}_2 - \mathbf{r}_1| \tag{2.30}$$

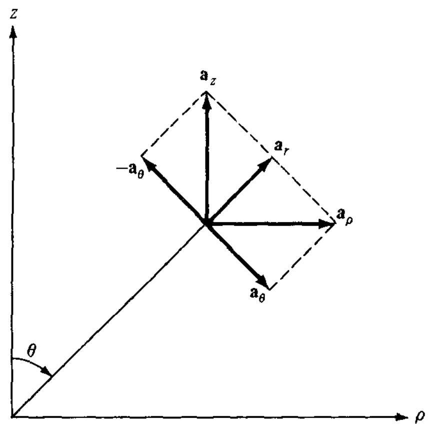

Figure 2.6 Unit vector transformations for cylindrical and spherical coordinates.

or

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \text{ (Cartesian)} \tag{2.31}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \text{ (cylindrical)} \tag{2.32}$$

$$\begin{aligned} d^2 = {} & r_2^2 + r_1^2 - 2r_1r_2 \cos\theta_2 \cos\theta_1 \\ & - 2r_1r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1) \text{ (spherical)} \end{aligned} \tag{2.33}$$

EXAMPLE 2.1

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2, y = 6, z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62°$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the Cartesian system, $\mathbf{A}$ at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector $\mathbf{A}$, $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

or

$$A_\rho = y\cos\phi + (x+z)\sin\phi$$
$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$
$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\begin{aligned} \mathbf{A} = (A_\rho, A_\phi, A_z) = {} & [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho \\ & + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi \end{aligned}$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$\begin{aligned} \mathbf{A} &= \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho \\ &\quad + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi \\ &= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi \end{aligned}$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

or

$$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$
$$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$
$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi)\\ &= r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r\\ &\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta\\ &\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi\end{aligned}$$

At P

$$r = 7, \qquad \tan\phi = \frac{6}{-2}, \qquad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}, \qquad \cos\theta = \frac{3}{7}, \qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\begin{aligned}\mathbf{A} &= 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_r\\ &\quad + 7\cdot\left[\frac{\sqrt{40}}{7}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\theta\\ &\quad + 7\cdot\left[\frac{-\sqrt{40}}{7}\cdot\frac{36}{40} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi\\ &= \frac{-6}{7}\mathbf{a}_r - \frac{18}{7\sqrt{40}}\mathbf{a}_\theta - \frac{38}{\sqrt{40}}\mathbf{a}_\phi\\ &= -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi\end{aligned}$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

PRACTICE EXERCISE 2.1

(a) Convert points $P(1, 3, 5)$, $T(0, -4, 3)$, and $S(-3, -4, -10)$ from Cartesian to cylindrical and spherical coordinates.

(b) Transform vector

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2}\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz\,\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

to cylindrical and spherical coordinates.

(c) Evaluate $\mathbf{Q}$ at T in the three coordinate systems.

Answer: (a) $P(3.162, 71.56°, 5)$, $P(5.916, 32.31°, 71.56°)$, $T(4, 270°, 3)$, $T(5, 53.13°, 270°)$, $S(5, 233.1°, -10)$, $S(11.18, 153.43°, 233.1°)$

(b) $\dfrac{\rho}{\sqrt{\rho^2 + z^2}}(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi - z\sin\phi\,\mathbf{a}_z)$, $\sin\theta\,(\sin\theta\cos\phi - r\cos^2\theta\sin\phi)\mathbf{a}_r + \sin\theta\cos\theta\,(\cos\phi + r\sin\theta\sin\phi)\mathbf{a}_\theta - \sin\theta\sin\phi\,\mathbf{a}_\phi$

(c) $0.8\mathbf{a}_x + 2.4\mathbf{a}_z$, $0.8\mathbf{a}_\phi + 2.4\mathbf{a}_z$, $1.44\mathbf{a}_r - 1.92\mathbf{a}_\theta + 0.8\mathbf{a}_\phi$.

EXAMPLE 2.2

Express vector

$$\mathbf{B} = \frac{10}{r}\,\mathbf{a}_r + r\cos\theta\,\mathbf{a}_\theta + \mathbf{a}_\phi$$

in Cartesian and cylindrical coordinates. Find $\mathbf{B}\,(-3, 4, 0)$ and $\mathbf{B}\,(5, \pi/2, -2)$.

Solution:

Using eq. (2.28):

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \dfrac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

or

$$B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$$

$$B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi$$

$$B_z = \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

But $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\dfrac{\sqrt{x^2 + y^2}}{z}$, and $\phi = \tan^{-1}\dfrac{y}{x}$

Hence,

$$\sin\theta = \frac{\rho}{r} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \qquad \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}, \qquad \cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}$$

Substituting all these gives

$$B_x = \frac{10\sqrt{x^2 + y^2}}{(x^2 + y^2 + z^2)} \cdot \frac{x}{\sqrt{x^2 + y^2}} + \frac{\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)} \cdot \frac{z^2 x}{\sqrt{x^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}}$$

$$= \frac{10x}{x^2 + y^2 + z^2} + \frac{xz^2}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} - \frac{y}{\sqrt{(x^2 + y^2)}}$$

$$B_y = \frac{10\sqrt{x^2 + y^2}}{(x^2 + y^2 + z^2)} \cdot \frac{y}{\sqrt{x^2 + y^2}} + \frac{\sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2} \cdot \frac{z^2 y}{\sqrt{x^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}}$$

$$= \frac{10y}{x^2 + y^2 + z^2} + \frac{yz^2}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} + \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_z = \frac{10z}{x^2 + y^2 + z^2} - \frac{z\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

where B_x, B_y, and B_z are as given above.

At $(-3, 4, 0)$, $x = -3$, $y = 4$, and $z = 0$, so

$$B_x = -\frac{30}{25} + 0 - \frac{4}{5} = -2$$

$$B_y = \frac{40}{25} + 0 - \frac{3}{5} = 1$$

$$B_z = 0 - 0 = 0$$

Thus,

$$\mathbf{B} = -2\mathbf{a}_x + \mathbf{a}_y$$

For spherical to cylindrical vector transformation (see Problem 2.9),

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

or

$$B_\rho = \frac{10}{r}\sin\theta + r\cos^2\theta$$

$$B_\phi = 1$$

$$B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta$$

But $r = \sqrt{\rho^2 + z^2}$ and $\theta = \tan^{-1}\dfrac{\rho}{z}$

Thus,

$$\sin\theta = \frac{\rho}{\sqrt{\rho^2 + z^2}}, \qquad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}}$$

$$B_\rho = \frac{10\rho}{\rho^2 + z^2} + \sqrt{\rho^2 + z^2} \cdot \frac{z^2}{\rho^2 + z^2}$$

$$B_z = \frac{10z}{\rho^2 + z^2} - \sqrt{\rho^2 + z^2} \cdot \frac{\rho z}{\rho^2 + z^2}$$

Hence,

$$\mathbf{B} = \left(\frac{10\rho}{\rho^2 + z^2} + \frac{z^2}{\sqrt{\rho^2 + z^2}}\right)\mathbf{a}_\rho + \mathbf{a}_\phi + \left(\frac{10z}{\rho^2 + z^2} - \frac{\rho z}{\sqrt{\rho^2 + z^2}}\right)\mathbf{a}_z$$

At $(5, \pi/2, -2)$, $\rho = 5$, $\phi = \pi/2$, and $z = -2$, so

$$\begin{aligned}\mathbf{B} &= \left(\frac{50}{29} + \frac{4}{\sqrt{29}}\right)\mathbf{a}_\rho + \mathbf{a}_\phi + \left(\frac{-20}{29} + \frac{10}{\sqrt{29}}\right)\mathbf{a}_z \\ &= 2.467\mathbf{a}_\rho + \mathbf{a}_\phi + 1.167\mathbf{a}_z\end{aligned}$$

Note that at $(-3, 4, 0)$,

$$|\mathbf{B}(x, y, z)| = |\mathbf{B}(\rho, \phi, z)| = |\mathbf{B}(r, \theta, \phi)| = 2.907$$

This may be used to check the correctness of the result whenever possible.

PRACTICE EXERCISE 2.2

Express the following vectors in Cartesian coordinates:

(a) $\mathbf{A} = \rho z \sin\phi\, \mathbf{a}_\rho + 3\rho\cos\phi\, \mathbf{a}_\phi + \rho\cos\phi\sin\phi\, \mathbf{a}_z$

(b) $\mathbf{B} = r^2\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\phi$

Answer: (a) $\mathbf{A} = \dfrac{1}{\sqrt{x^2 + y^2}}[(xyz - 3xy)\, \mathbf{a}_x + (zy^2 + 3x^2)\, \mathbf{a}_y + xy\, \mathbf{a}_z]$

(b) $\mathbf{B} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}\{[x(x^2 + y^2 + z^2) - y]\mathbf{a}_x$
$+ [y(x^2 + y^2 + z^2) + x]\mathbf{a}_y + z(x^2 + y^2 + z^2)\mathbf{a}_z\}$

2.5 CONSTANT-COORDINATE SURFACES

Surfaces in Cartesian, cylindrical, or spherical coordinate systems are easily generated by keeping one of the coordinate variables constant and allowing the other two to vary. In the

Cartesian system, if we keep x constant and allow y and z to vary, an infinite plane is generated. Thus we could have infinite planes

$$\begin{aligned} x &= \text{constant} \\ y &= \text{constant} \\ z &= \text{constant} \end{aligned} \tag{2.34}$$

which are perpendicular to the x-, y-, and z-axes, respectively, as shown in Figure 2.7. The intersection of two planes is a line. For example,

$$x = \text{constant}, \qquad y = \text{constant} \tag{2.35}$$

is the line RPQ parallel to the z-axis. The intersection of three planes is a point. For example,

$$x = \text{constant}, \qquad y = \text{constant}, \qquad z = \text{constant} \tag{2.36}$$

is the point $P(x, y, z)$. Thus we may define point P as the intersection of three orthogonal infinite planes. If P is $(1, -5, 3)$, then P is the intersection of planes $x = 1$, $y = -5$, and $z = 3$.

Orthogonal surfaces in cylindrical coordinates can likewise be generated. The surfaces

$$\begin{aligned} \rho &= \text{constant} \\ \phi &= \text{constant} \\ z &= \text{constant} \end{aligned} \tag{2.37}$$

are illustrated in Figure 2.8, where it is easy to observe that ρ = constant is a circular cylinder, ϕ = constant is a semiinfinite plane with its edge along the z-axis, and z = constant is the same infinite plane as in a Cartesian system. Where two surfaces meet is either a line or a circle. Thus,

$$z = \text{constant}, \qquad \rho = \text{constant} \tag{2.38}$$

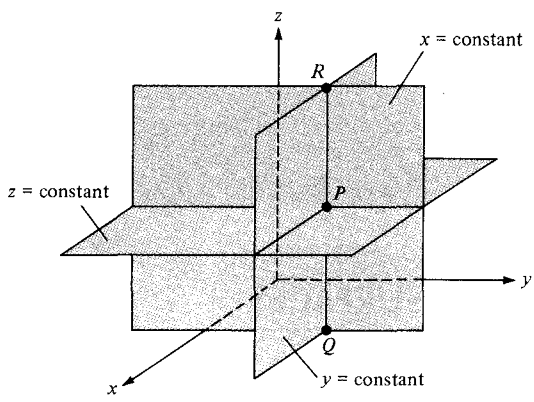

Figure 2.7 Constant x, y, and z surfaces.

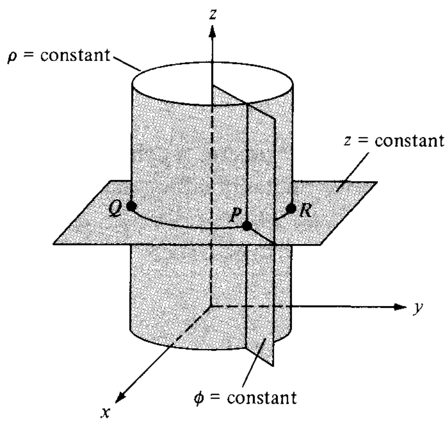

Figure 2.8 Constant ρ, ϕ, and z surfaces.

is a circle QPR of radius ρ, whereas z = constant, ϕ = constant is a semiinfinite line. A point is an intersection of the three surfaces in eq. (2.37). Thus,

$$\rho = 2, \qquad \phi = 60°, \qquad z = 5 \tag{2.39}$$

is the point $P(2, 60°, 5)$.

The orthogonal nature of the spherical coordinate system is evident by considering the three surfaces

$$\begin{aligned} r &= \text{constant} \\ \theta &= \text{constant} \\ \phi &= \text{constant} \end{aligned} \tag{2.40}$$

which are shown in Figure 2.9, where we notice that r = constant is a sphere with its center at the origin; θ = constant is a circular cone with the z-axis as its axis and the origin as its vertex; ϕ = constant is the semiinfinite plane as in a cylindrical system. A line is formed by the intersection of two surfaces. For example:

$$r = \text{constant}, \qquad \phi = \text{constant} \tag{2.41}$$

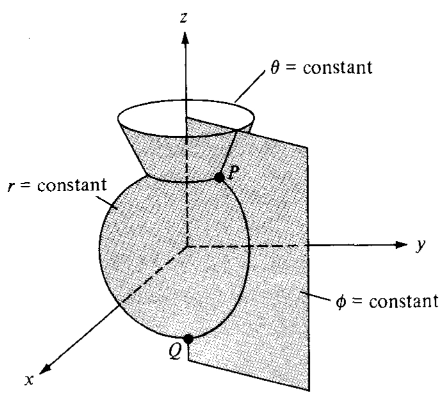

Figure 2.9 Constant r, θ, and ϕ surfaces.

is a semicircle passing through Q and P. The intersection of three surfaces gives a point. Thus,

$$r = 5, \qquad \theta = 30°, \qquad \phi = 60° \tag{2.42}$$

is the point $P(5, 30°, 60°)$. We notice that in general, a point in three-dimensional space can be identified as the intersection of three mutually orthogonal surfaces. Also, a unit normal vector to the surface n = constant is $\pm \mathbf{a}_n$, where n is x, y, z, ρ, ϕ, r, or θ. For example, to plane $x = 5$, a unit normal vector is $\pm \mathbf{a}_x$ and to plane $\phi = 20°$, a unit normal vector is $\mathbf{a}_\phi$.

EXAMPLE 2.3

Two uniform vector fields are given by $\mathbf{E} = -5\mathbf{a}_\rho + 10\mathbf{a}_\phi + 3\mathbf{a}_z$ and $\mathbf{F} = \mathbf{a}_\rho + 2\mathbf{a}_\phi - 6\mathbf{a}_z$. Calculate

(a) $|\mathbf{E} \times \mathbf{F}|$

(b) The vector component of $\mathbf{E}$ at $P(5, \pi/2, 3)$ parallel to the line $x = 2, z = 3$

(c) The angle $\mathbf{E}$ makes with the surface $z = 3$ at P

Solution:

(a)
$$\begin{aligned}\mathbf{E} \times \mathbf{F} &= \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ -5 & 10 & 3 \\ 1 & 2 & -6 \end{vmatrix} \\ &= (-60 - 6)\mathbf{a}_\rho + (3 - 30)\mathbf{a}_\phi + (-10 - 10)\mathbf{a}_z \\ &= (-66, -27, -20)\end{aligned}$$

$$|\mathbf{E} \times \mathbf{F}| = \sqrt{66^2 + 27^2 + 20^2} = 74.06$$

(b) Line $x = 2, z = 3$ is parallel to the y-axis, so the component of $\mathbf{E}$ parallel to the given line is

$$(\mathbf{E} \cdot \mathbf{a}_y)\mathbf{a}_y$$

But at $P(5, \pi/2, 3)$

$$\begin{aligned}\mathbf{a}_y &= \sin\phi\, \mathbf{a}_\rho + \cos\phi\, \mathbf{a}_\phi \\ &= \sin\pi/2\, \mathbf{a}_\rho + \cos\pi/2\, \mathbf{a}_\phi = \mathbf{a}_\rho\end{aligned}$$

Therefore,

$$(\mathbf{E} \cdot \mathbf{a}_y)\mathbf{a}_y = (\mathbf{E} \cdot \mathbf{a}_\rho)\mathbf{a}_\rho = -5\mathbf{a}_\rho \qquad (\text{or } -5\mathbf{a}_y)$$

(c) Utilizing the fact that the z-axis is normal to the surface $z = 3$, the angle between the z-axis and $\mathbf{E}$, as shown in Figure 2.10, can be found using the dot product:

$$\mathbf{E} \cdot \mathbf{a}_z = |\mathrm{E}|(1)\cos\theta_{Ez} \rightarrow 3 = \sqrt{134}\cos\theta_{Ez}$$

$$\cos\theta_{Ez} = \frac{3}{\sqrt{134}} = 0.2592 \rightarrow \theta_{Ez} = 74.98°$$

Hence, the angle between $z = 3$ and $\mathbf{E}$ is

$$90° - \theta_{Ez} = 15.02°$$

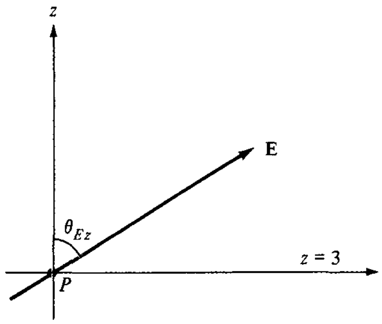

Figure 2.10 For Example 2.3(c).

PRACTICE EXERCISE 2.3

Given the vector field

$$\mathbf{H} = \rho z \cos\phi \, \mathbf{a}_\rho + e^{-2} \sin\frac{\phi}{2} \, \mathbf{a}_\phi + \rho^2 \mathbf{a}_z$$

At point $(1, \pi/3, 0)$, find

(a) $\mathbf{H} \cdot \mathbf{a}_x$
(b) $\mathbf{H} \times \mathbf{a}_\theta$
(c) The vector component of **H** normal to surface $\rho = 1$
(d) The scalar component of **H** tangential to the plane $z = 0$

Answer: (a) -0.433, (b) $-0.5 \, \mathbf{a}_\rho$, (c) $0 \, \mathbf{a}_\rho$, (d) 0.5.

EXAMPLE 2.4

Given a vector field

$$\mathbf{D} = r \sin\phi \, \mathbf{a}_r - \frac{1}{r} \sin\theta \cos\phi \, \mathbf{a}_\theta + r^2 \mathbf{a}_\phi$$

determine

(a) **D** at $P(10, 150°, 330°)$
(b) The component of **D** tangential to the spherical surface $r = 10$ at P
(c) A unit vector at P perpendicular to **D** and tangential to the cone $\theta = 150°$

Solution:

(a) At P, $r = 10$, $\theta = 150°$, and $\phi = 330°$. Hence,

$$\mathbf{D} = 10 \sin 330° \, \mathbf{a}_r - \frac{1}{10} \sin 150° \cos 330° \, \mathbf{a}_\theta + 100 \, \mathbf{a}_\phi = (-5, 0.043, 100)$$

(b) Any vector $\mathbf{D}$ can always be resolved into two orthogonal components:

$$\mathbf{D} = \mathbf{D}_t + \mathbf{D}_n$$

where D_t is tangential to a given surface and $\mathbf{D}_n$ is normal to it. In our case, since $\mathbf{a}_r$ is normal to the surface $r = 10$,

$$\mathbf{D}_n = r \sin \phi \, \mathbf{a}_r = -5\mathbf{a}_r$$

Hence,

$$\mathbf{D}_t = \mathbf{D} - \mathbf{D}_n = 0.043\mathbf{a}_\theta + 100\mathbf{a}_\phi$$

(c) A vector at P perpendicular to $\mathbf{D}$ and tangential to the cone $\theta = 150°$ is the same as the vector perpendicular to both $\mathbf{D}$ and $\mathbf{a}_\theta$. Hence,

$$\mathbf{D} \times \mathbf{a}_\theta = \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\theta & \mathbf{a}_\phi \\ -5 & 0.043 & 100 \\ 0 & 1 & 0 \end{vmatrix} = -100\mathbf{a}_r - 5\mathbf{a}_\phi$$

A unit vector along this is

$$\mathbf{a} = \frac{-100\mathbf{a}_r - 5\mathbf{a}_\phi}{\sqrt{100^2 + 5^2}} = -0.9988\mathbf{a}_r - 0.0499\mathbf{a}_\phi$$

PRACTICE EXERCISE 2.4

If $\mathbf{A} = 3\mathbf{a}_r + 2\mathbf{a}_\theta - 6\mathbf{a}_\phi$ and $\mathbf{B} = 4\mathbf{a}_r + 3\mathbf{a}_\phi$, determine

(a) $\mathbf{A} \cdot \mathbf{B}$

(b) $|\mathbf{A} \times \mathbf{B}|$

(c) The vector component of $\mathbf{A}$ along $\mathbf{a}_z$ at $(1, \pi/3, 5\pi/4)$

Answer: (a) -6, (b) 34.48, (c) $-0.116\mathbf{a}_r + 0.201\mathbf{a}_\theta$.

SUMMARY

1. The three common coordinate systems we shall use throughout the text are the Cartesian (or rectangular), the circular cylindrical, and the spherical.
2. A point P is represented as $P(x, y, z)$, $P(\rho, \phi, z)$, and $P(r, \theta, \phi)$ in the Cartesian, cylindrical, and spherical systems respectively. A vector field $\mathbf{A}$ is represented as (A_x, A_y, A_z) or $A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ in the Cartesian system, as (A_ρ, A_ϕ, A_z) or $A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$ in the cylindrical system, and as (A_r, A_θ, A_ϕ) or $A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$ in the spherical system. It is preferable that mathematical operations (addition, subtraction, product, etc.) be performed in the same coordinate system. Thus, point and vector transformations should be performed whenever necessary.
3. Fixing one space variable defines a surface; fixing two defines a line; fixing three defines a point.
4. A unit normal vector to surface n = constant is $\pm\, \mathbf{a}_n$.

REVIEW QUESTIONS

2.1 The ranges of θ and ϕ as given by eq. (2.17) are not the only possible ones. The following are all alternative ranges of θ and ϕ, except

(a) $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$
(b) $0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi$
(c) $-\pi \leq \theta \leq \pi, 0 \leq \phi \leq \pi$
(d) $-\pi/2 \leq \theta \leq \pi/2, 0 \leq \phi < 2\pi$
(e) $0 \leq \theta \leq \pi, \quad -\pi \leq \phi < \pi$
(f) $-\pi \leq \theta < \pi, \quad -\pi \leq \phi < \pi$

2.2 At Cartesian point $(-3, 4, -1)$, which of these is incorrect?

(a) $\rho = -5$
(b) $r = \sqrt{26}$
(c) $\theta = \tan^{-1} \dfrac{5}{-1}$
(d) $\phi = \tan^{-1} \dfrac{4}{-3}$

2.3 Which of these is not valid at point (0, 4, 0)?

(a) $\mathbf{a}_\phi = -\mathbf{a}_x$
(b) $\mathbf{a}_\theta = -\mathbf{a}_z$
(c) $\mathbf{a}_r = 4\mathbf{a}_y$
(d) $\mathbf{a}_\rho = \mathbf{a}_y$

2.4 A unit normal vector to the cone $\theta = 30°$ is:

(a) $\mathbf{a}_r$
(b) $\mathbf{a}_\theta$
(c) $\mathbf{a}_\phi$
(d) none of the above

2.5 At every point in space, $\mathbf{a}_\phi \cdot \mathbf{a}_\theta = 1$.

(a) True
(b) False

2.6 If $\mathbf{H} = 4\mathbf{a}_\rho - 3\mathbf{a}_\phi + 5\mathbf{a}_z$, at $(1, \pi/2, 0)$ the component of $\mathbf{H}$ parallel to surface $\rho = 1$ is

(a) $4\mathbf{a}_\rho$
(b) $5\mathbf{a}_z$
(c) $-3\mathbf{a}_\phi$
(d) $-3\mathbf{a}_\phi + 5\mathbf{a}_z$
(e) $5\mathbf{a}_\phi + 3\mathbf{a}_z$

2.7 Given $\mathbf{G} = 20\mathbf{a}_r + 50\mathbf{a}_\theta + 40\mathbf{a}_\phi$, at $(1, \pi/2, \pi/6)$ the component of $\mathbf{G}$ perpendicular to surface $\theta = \pi/2$ is

(a) $20\mathbf{a}_r$
(b) $50\mathbf{a}_\theta$
(c) $40\mathbf{a}_\phi$
(d) $20\mathbf{a}_r + 40\mathbf{a}_\theta$
(e) $-40\mathbf{a}_r + 20\mathbf{a}_\phi$

2.8 Where surfaces $\rho = 2$ and $z = 1$ intersect is

(a) an infinite plane
(b) a semiinfinite plane
(c) a circle
(d) a cylinder
(e) a cone

2.9 Match the items in the left list with those in the right list. Each answer can be used once, more than once, or not at all.

(a) $\theta = \pi/4$	(i) infinite plane
(b) $\phi = 2\pi/3$	(ii) semiinfinite plane
(c) $x = -10$	(iii) circle
(d) $r = 1, \theta = \pi/3, \phi = \pi/2$	(iv) semicircle
(e) $\rho = 5$	(v) straight line
(f) $\rho = 3, \phi = 5\pi/3$	(vi) cone
(g) $\rho = 10, z = 1$	(vii) cylinder
(h) $r = 4, \phi = \pi/6$	(viii) sphere
(i) $r = 5, \theta = \pi/3$	(ix) cube
	(x) point

2.10 A wedge is described by $z = 0, 30° < \phi < 60°$. Which of the following is incorrect:

(a) The wedge lies in the $x - y$ plane.
(b) It is infinitely long
(c) On the wedge, $0 < \rho < \infty$
(d) A unit normal to the wedge is $\pm \mathbf{a}_z$
(e) The wedge includes neither the x-axis nor the y-axis

Answers: 2.1b,f, 2.2a, 2.3c, 2.4b, 2.5b, 2.6d, 2.7b, 2.8c, 2.9a-(vi), b-(ii), c-(i), d-(x), e-(vii), f-(v), g-(iii), h-(iv), i-(iii), 2.10b.

PROBLEMS

2.1 Express the following points in Cartesian coordinates:

(a) $P(1, 60°, 2)$
(b) $Q(2, 90°, -4)$
(c) $R(, 45°, 210°)$
(d) $T(4, \pi/2, \pi/6)$

2.2 Express the following points in cylindrical and spherical coordinates:

(a) $P(1, -4, -3)$
(b) $Q(3, 0, 5)$
(c) $R(-2, 6, 0)$

2.3 (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.
(b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.

2.4 Transform the following vectors to cylindrical and spherical coordinates:

(a) $\mathbf{D} = (x + z)\mathbf{a}_y$
(b) $\mathbf{E} = (y^2 - x^2)\mathbf{a}_x + xyz\mathbf{a}_y + (x^2 - z^2)\mathbf{a}_z$

2.5 Convert the following vectors to cylindrical and spherical systems:

(a) $\mathbf{F} = \dfrac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$

(b) $\mathbf{G} = (x^2 + y^2)\left[\dfrac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}\right]$

2.6 Express the following vectors in Cartesian coordinates:

(a) $\mathbf{A} = \rho(z^2 + 1)\mathbf{a}_\rho - \rho z \cos\phi\, \mathbf{a}_\phi$
(b) $\mathbf{B} = 2r \sin\theta \cos\phi\, \mathbf{a}_r + r \cos\theta \cos\theta\, \mathbf{a}_\theta - r \sin\phi\, \mathbf{a}_\phi$

2.7 Convert the following vectors to Cartesian coordinates:

(a) $\mathbf{C} = z \sin\phi\, \mathbf{a}_\rho - \rho \cos\phi\, \mathbf{a}_\phi + 2\rho z\mathbf{a}_z$

(b) $\mathbf{D} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r + \dfrac{\cos\theta}{r^2}\mathbf{a}_\theta$

2.8 Prove the following:

(a)
$$\begin{aligned}
\mathbf{a}_x \cdot \mathbf{a}_\rho &= \cos\phi \\
\mathbf{a}_x \cdot \mathbf{a}_\phi &= -\sin\phi \\
\mathbf{a}_y \cdot \mathbf{a}_\rho &= \sin\phi \\
\mathbf{a}_y \cdot \mathbf{a}_\phi &= \cos\phi
\end{aligned}$$

(b)
$$\begin{aligned}
\mathbf{a}_x \cdot \mathbf{a}_r &= \sin\theta \cos\phi \\
\mathbf{a}_x \cdot \mathbf{a}_\theta &= \cos\theta \cos\phi \\
\mathbf{a}_y \cdot \mathbf{a}_r &= \sin\theta \sin\phi
\end{aligned}$$

$$\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta \sin\phi$$
$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$$

2.9 (a) Show that point transformation between cylindrical and spherical coordinates is obtained using

$$r = \sqrt{\rho^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\rho}{z}, \qquad \phi = \phi$$

or

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \qquad \phi = \phi$$

(b) Show that vector transformation between cylindrical and spherical coordinates is obtained using

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(*Hint:* Make use of Figures 2.5 and 2.6.)

2.10 (a) Express the vector field

$$\mathbf{H} = xy^2z\mathbf{a}_x + x^2yz\mathbf{a}_y + xyz^2\mathbf{a}_z$$

in cylindrical and spherical coordinates.

(b) In both cylindrical and spherical coordinates, determine **H** at $(3, -4, 5)$.

2.11 Let $\mathbf{A} = \rho\cos\theta\,\mathbf{a}_\rho + \rho z^2 \sin\phi\,\mathbf{a}_z$

(a) Transform **A** into rectangular coordinates and calculate its magnitude at point $(3, -4, 0)$.

(b) Transform **A** into spherical system and calculate its magnitude at point $(3, -4, 0)$.

2.12 The transformation $(A_\rho, A_\phi, A_z) \to (A_x, A_y, A_z)$ in eq. (2.15) is not complete. Complete it by expressing $\cos\phi$ and $\sin\phi$ in terms of x, y, and z. Do the same thing to the transformation $(A_r, A_\theta, A_\phi) \to (A_x, A_y, A_z)$ in eq. (2.28).

2.13 In Practice Exercise 2.2, express **A** in spherical and **B** in cylindrical coordinates. Evaluate **A** at $(10, \pi/2, 3\pi/4)$ and **B** at $(2, \pi/6, 1)$.

2.14 Calculate the distance between the following pairs of points:

(a) $(2, 1, 5)$ and $(6, -1, 2)$
(b) $(3, \pi/2, -1)$ and $(5, 3\pi/2, 5)$
(c) $(10, \pi/4, 3\pi/4)$ and $(5, \pi/6, 7\pi/4)$.

2.15 Describe the intersection of the following surfaces:

(a) $x = 2, \quad y = 5$
(b) $x = 2, \quad y = -1, \quad z = 10$
(c) $r = 10, \quad \theta = 30°$
(d) $\rho = 5, \quad \phi = 40°$
(e) $\phi = 60°, \quad z = 10$
(f) $r = 5, \quad \phi = 90°$

2.16 At point $T(2, 3, -4)$, express $\mathbf{a}_z$ in the spherical system and $\mathbf{a}_r$ in the rectangular system.

***2.17** Given vectors $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{B} = -5\mathbf{a}_\rho + \mathbf{a}_\phi - 3\mathbf{a}_z$, find

(a) $\mathbf{A} + \mathbf{B}$ at $P(0, 2, -5)$
(b) The angle between $\mathbf{A}$ and $\mathbf{B}$ at P
(c) The scalar component of $\mathbf{A}$ along $\mathbf{B}$ at P

2.18 Given that $\mathbf{G} = (x + y^2)\mathbf{a}_x + xz\mathbf{a}_y + (z^2 + zy)\mathbf{a}_z$, find the vector component of $\mathbf{G}$ along $\mathbf{a}_\phi$ at point $P(8, 30°, 60°)$. Your answer should be left in the Cartesian system.

***2.19** If $\mathbf{J} = r \sin\theta \cos\phi \, \mathbf{a}_r - \cos 2\theta \sin\phi \, \mathbf{a}_\theta + \tan\dfrac{\theta}{2} \ln r \, \mathbf{a}_\phi$ at $T(2, \pi/2, 3\pi/2)$, determine the vector component of $\mathbf{J}$ that is

(a) Parallel to $\mathbf{a}_z$
(b) Normal to surface $\phi = 3\pi/2$
(c) Tangential to the spherical surface $r = 2$
(d) Parallel to the line $y = -2, z = 0$

2.20 Let $\mathbf{H} - 5\rho \sin\phi \, \mathbf{a}_\rho - \rho z \cos\phi \, \mathbf{a}_\phi + 2\rho\mathbf{a}_z$. At point $P(2, 30°, -1)$, find:

(a) a unit vector along $\mathbf{H}$
(b) the component of $\mathbf{H}$ parallel to $\mathbf{a}_x$
(c) the component of $\mathbf{H}$ normal to $\rho = 2$
(d) the component of $\mathbf{H}$ tangential to $\phi = 30°$

***2.21** Let

$$\mathbf{A} = \rho(z^2 - 1)\mathbf{a}_\rho - \rho z \cos\phi\ \mathbf{a}_\phi + \rho^2 z^2 \mathbf{a}_z$$

and

$$\mathbf{B} = r^2 \cos\phi\ \mathbf{a}_r + 2r \sin\theta\ \mathbf{a}_\phi$$

At $T(-3, 4, 1)$, calculate: (a) **A** and **B**, (b) the vector component in cylindrical coordinates of **A** along **B** at T, (c) the unit vector in spherical coordinates perpendicular to both **A** and **B** at T.

***2.22** Another way of defining a point P in space is $(r, \alpha, \beta, \gamma)$ where the variables are portrayed in Figure 2.11. Using this definition, find $(r, \alpha, \beta, \gamma)$ for the following points:

(a) $(-2, 3, 6)$

(b) $(4, 30°, -3)$

(c) $(3, 30°, 60°)$

(*Hint:* r is the spherical r, $0 \leq \alpha, \beta, \gamma < 2\pi$.)

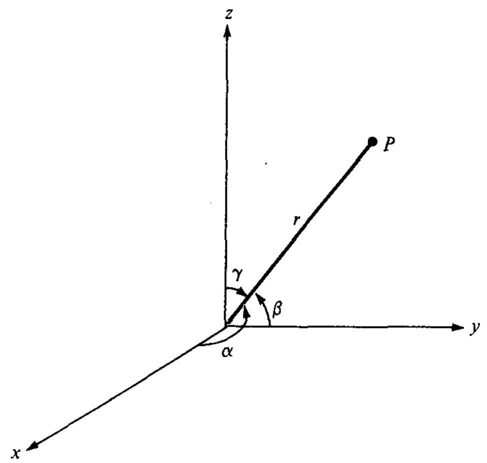

Figure 2.11 For Problem 2.22.

2.23 A vector field in "mixed" coordinate variables is given by

$$\mathbf{G} = \frac{x\cos\phi}{\rho}\mathbf{a}_x + \frac{2yz}{\rho^2} + \left(1 - \frac{x^2}{\rho^2}\right)\mathbf{a}_z$$

Express **G** completely in spherical system.