Chapter one
Harmonic oscillators

1.1: Introduction:

In every daily life we come a cross numerous things that move. These motions are of two types;

The motion in which the body moves about a mean position i.e. a fixed point.

The motion in which the body moves from one place to the other with respect of time.

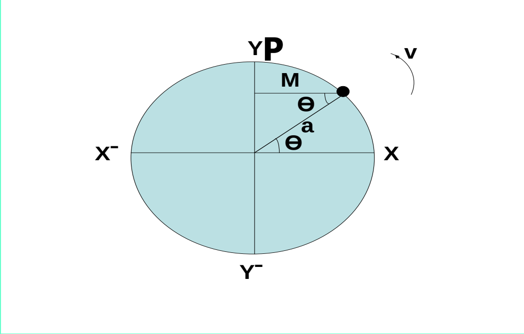
The first type of motion is called oscillatory motion (example; oscillating pendulum, vibration of a stretched string , movement of water in a cup , vibration of electron , movement of light in a laser beam etc.)

A moving train, flying aero plane, moving ball etc. correspond to the second type of motion.

Sometimes both the types of motion are exhibited in the same phenomenon depending on our point of view.

1.2: Simple harmonic motion:

Let P be a particle moving on the circumference of a circle of radius a with a uniform velocity v



Let v=aω

a is radius ω is angular velocity

(The circle along which **P** moves is called

the circle of reference.)

* **(As the particle P moves round the circle continuously with uniform velocity v , the foot of the Perpendicular M, vibrates along the diameter. If the motion of P is uniform, then the motion of M is periodic (i.e., it takes the same time to vibrate once between the points y and y´))**

At any instant the distance of M from the center O of the circle is called the **displacement**.

* **If the particle moved from** X **to** p **in the time** t, then

LPOX= LMPO = θ = ωt

From the ΔMPO

$$\sin\theta = \sin wt = \frac{OM}{a}$$

$$OM = y = a\sin wt$$

OM is called the displaceme

- **(The displacement of a vibrating particle at any instant can be defined as its distance from the mean position of rest.)**

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Amplitude; **the maximum displacements of vibrating particle is called its** **amplitude.**

$$\text{Displacement} = y = a\sin wt$$

- **The rate of change of displacement is called the velocity of the vibrating particle.**

$$Velocity = \frac{dy}{dt} = aw\cos wt$$

*** The rate of change of velocity is called its acceleration.**

A

$$\text{Acceleration} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = -aw^2\sin wt = -w^2y$$

Angle	Position of the vibrating particle	Displacement	Velocity	Acceleration
wt	M	Y = a sin wt	dy/dt = aw coswt	$d^2y/dt^2 = -aw^2 \sin wt$
0	O	0	+aw	0
π/2	Y	+a	0	$-aw^2$
π	O	0	-aw	0
3 π/2	ȳ	-a	0	$+aw^2$
2 π	O	0	+aw	0

Oscillatory behavior;

The process repeats itself periodically. Thus the system Oscillates. In this process, Y, dy/dt, d^2y/dt^2 Continusly change with respect to time.

[Thus, the velocity of the vibrating particle is maximum at the mean position of rest and zero at the maximum position of vibration.

The acceleration of the vibrating particle is zero at the mean position of rest and maximum at the maximum position of vibration. The acceleration is always directed towards the mean position of rest and is directly proportional to the displacement of the vibrating particle.]

(This type of motion where the acceleration is directed towards a fixed point (the mean position of rest) and is proportional to the displacement of the vibrating particle is called simple harmonic motion)

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -w^2 y = -w^2 \times displacement$$

Numerically

$$w^2 = \frac{Acceleration}{Displacement}$$

$$w = 2\pi n = \sqrt{\frac{Acceleration}{Displacement}} = \frac{2\pi}{T}$$

$$T = 2\pi\sqrt{\frac{Displacement}{Acceleration}} = 2\pi\sqrt{K}$$

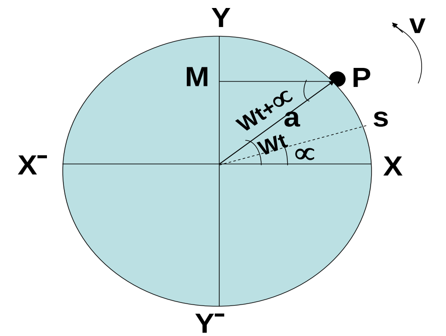
The time period of a particle vibrating simple harmonically
Where K is the displacement per unit acceleration

If the particle P revolves round the circle, n times per second, then the angular velocity w is given by;

$$n = \frac{1}{T}$$

Where T is the time period

$$w = 2\pi n = \frac{2\pi}{T}$$



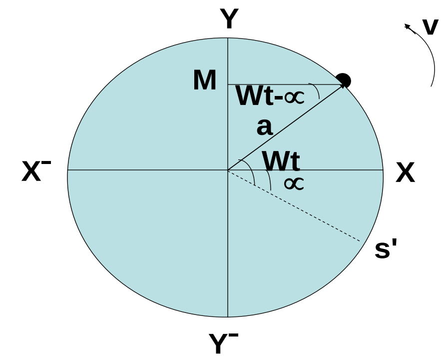
On the other hand, if the time is counted From the instant P is at S (∠sox=∝)Then the displacement;

$$y = a\sin 2\pi nt = a\sin 2\pi\frac{t}{T}$$

If the time is counted from the instant P is at then;

$$y = a\sin(wt + \alpha) = a\sin(2\pi\frac{t}{T} + \alpha)$$

$$y = a\sin(wt - \alpha) = a\sin(2\pi\frac{t}{T} - \alpha)$$



Phase of the vibrating particle:

1-The phase of a vibrating particle is defined as the ratio of the displacement of the vibrating article at any instant to the amplitude of the vibrating particle (y/a).

2-It is also equal to the angle by the radius vector since the vibrating particle last crossed its mean position of rest e.g., in the above equations wt, (wt+∝) or (wt-∝) are called phase angle.

The initial phase angle when t=0 is called the epoch.

Thus ∝ is called the epoch in the above expressions.

1.3: Differential Equation of SHM;

For a particle vibrating simple harmonically, the general equation of displacement is,

$$y = a\sin(wt + \alpha) - - - - (1)$$

Where y is displacement, is amplitude and ∝ is epoch of the vibrating particle.

$$\frac{dy}{dt} = a\,w\cos(wt + \alpha) - - - - (2)$$

$$\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -a\,w^2\sin(wt + \alpha) - - - - (3)$$

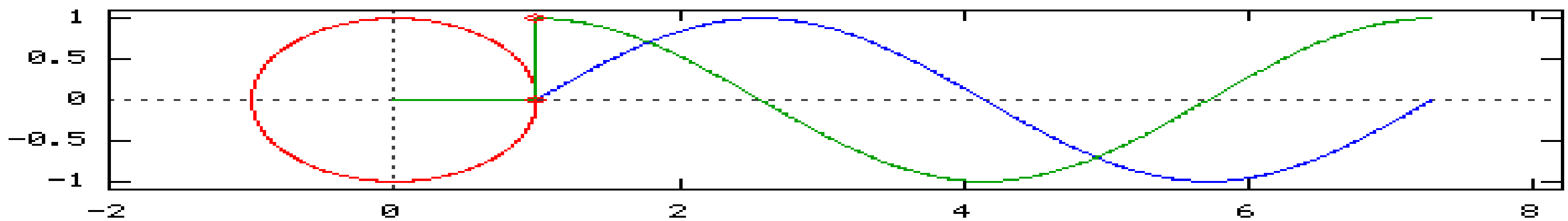
$$y = a\sin(wt + \alpha)$$

$$\therefore \frac{d^2y}{dt^2} = -w^2 y$$

But

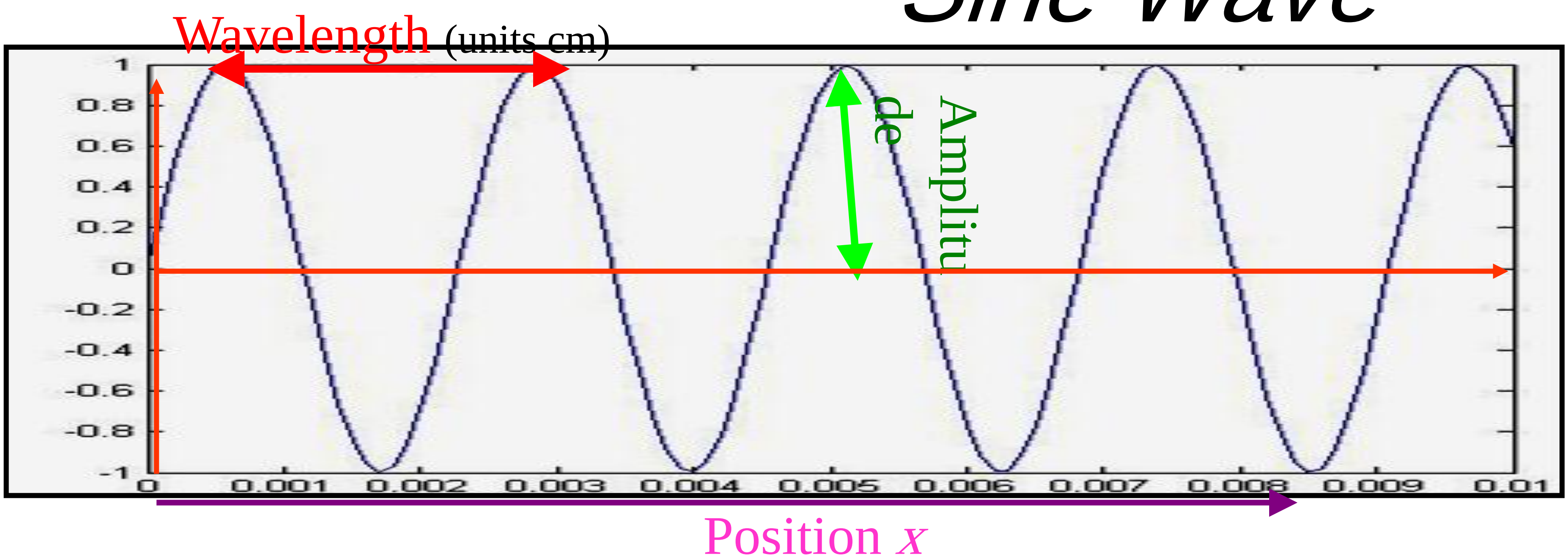
$$\therefore \frac{d^2y}{dt^2} + w^2 y = 0 - - - - - (3)$$

$$\frac{d^2y}{dt^2}$$



1
0.5
0
−0.5
−1
−2
0
2
4
6
8

Sine Wave



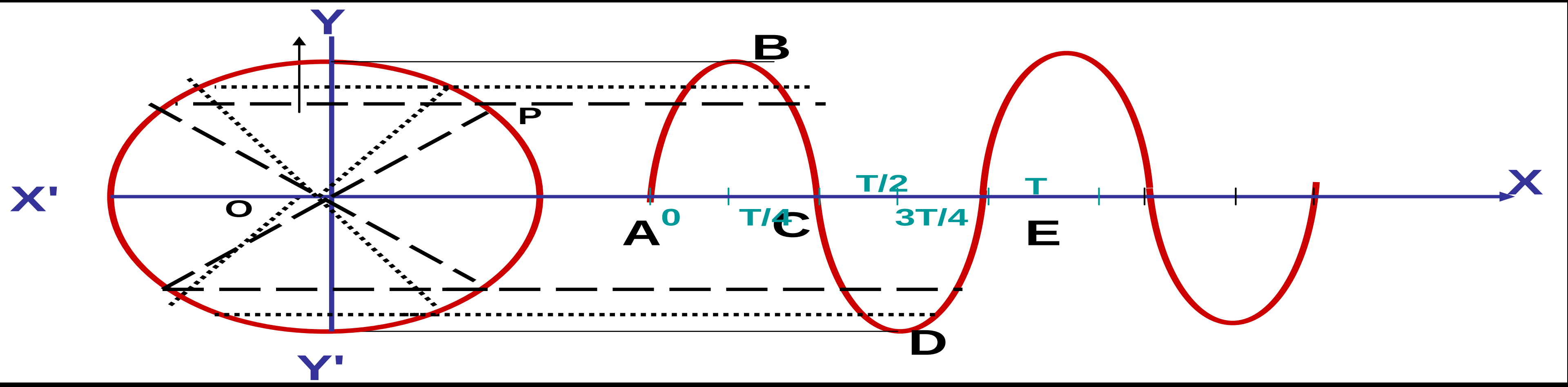
Wavelength (units cm)
Amplitude
Position *x*
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01

$$y = a\sin(wt + \alpha)$$

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{Displacement}{Acceleration}}$$

1.4: Graphical Representation of SHM;

$$y = a\sin(wt) = a\sin(2\pi\frac{t}{T})$$



The velocity of a particle moving with SHM is

$$v = \frac{dy}{dt} = aw\cos(wt)$$

The velocity – time graph is shown in figure (*) It is cosine curve.

$$\frac{d^2y}{dt^2} = -aw^2\sin(wt)$$

1.5: Average Kinetic Energy of a Vibrating Particle:-

$$y = a\sin(wt+\alpha)$$

$$v = \frac{dy}{dt} = a\,w\cos(wt+\alpha)$$

$$K.E = \frac{1}{2}mv^2$$

$$K.E = \frac{1}{2}ma^2w^2\cos^2(wt+\alpha)$$

$$\text{Average K.E} = \frac{1}{\text{T}}\int_0^T (K.E)\,dt = \frac{1}{\text{T}}\int_0^T \frac{1}{2}ma^2w^2\cos^2(wt+\alpha)\,dt$$

$$\text{Average K.E} = \frac{1}{\text{T}}\frac{ma^2w^2}{4}\int_0^T 2\cos^2(wt+\alpha)\,dt$$

$$\text{Average K.E} = \frac{1}{\text{T}}\frac{ma^2w^2}{4}\int_0^T (1+\cos 2(wt+\alpha)\,dt$$

$$\text{Average K.E} = \frac{1}{\text{T}}\frac{ma^2w^2}{4}\left(\int_0^T dt + \int_0^T \cos 2(wt+\alpha)\,dt\right)$$

$$\int_0^\pi \cos 2(wt+\alpha)\,dt = 0$$

$$\therefore \text{Average K.E} = \frac{1}{\text{T}}\frac{ma^2w^2}{4}(T+0)$$

$$\therefore \text{Average K.E} = \frac{ma^2 4\pi^2 n^2}{4}$$

$$\therefore \text{Average K.E} = \pi^2 ma^2 n^2$$

1.6: Total Energy of Vibrating Particle;

$$y = a\sin(wt+\alpha)$$

$$\sin(wt+\alpha) = \frac{y}{a}$$

$$\sin^2(wt+\alpha) = \frac{y^2}{a^2}$$

$$\because \sin^2(wt+\alpha) + \cos^2(wt+\alpha) = 1$$

$$\sin^2(wt+\alpha) = 1 - \cos^2(wt+\alpha)$$

$$\therefore 1 - \cos^2(wt+\alpha) = \frac{y^2}{a^2}$$

$$\cos^2(wt+\alpha) = 1 - \frac{y^2}{a^2}$$

$$\cos(wt+\alpha) = \sqrt{1-\frac{y^2}{a^2}} = \sqrt{\frac{a^2-y^2}{a^2}}$$

$$\cos(wt+\alpha) = \frac{\sqrt{a^2-y^2}}{a}$$

$$v = \frac{dy}{dt} = a\,w\cos(wt+\alpha) = aw\frac{\sqrt{a^2-y^2}}{a} = w\sqrt{a^2-y^2}$$

$$v = \frac{dy}{dt} = w\sqrt{a^2-y^2}$$

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}mw^2(a^2-y^2)$$

1.7: Energy of Vibration:

Work done = Force x displacement

$$y = a\sin(wt - \phi)$$

Let the periodic force be;

F=F$_o$ sinwt

$$w = \int F\,dy$$

$$w = \int F_o \sin wt\,\big(a\cos(wt - \phi)\big)d(wt)$$

The Total Work done:

$$y = a\sin(wt - \phi) \longrightarrow dy = a\cos(wt - \phi)\,d(wt)$$

$$w = aF_o \int_0^{2\pi} \sin wt\ \cos(wt - \phi)d(wt)$$

Since,

$$\cos(wt - \phi) = \cos wt\cos\phi + \sin wt\sin\phi$$

Work done per cycle of motion:

$$w = aF_o \int_0^{2\pi} \sin wt\ (\cos wt\cos\phi + \sin wt\sin\phi)d(wt)$$

Bu

$$w = aF_o\cos\phi \int_0^{2\pi} \sin wt\ \cos wt d(wt) + aF_o\sin\phi \int_0^{2\pi} \sin^2 wt\ d(wt)$$

Th

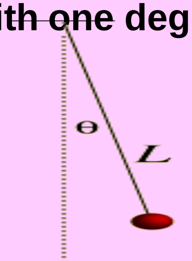
$$w = aF_o\cos\phi\left(\frac{\sin 2wt}{2}\right)\Big|_0^{2\pi} + aF_o\sin\phi\left(\frac{wt}{2} - \frac{\sin 2wt}{4}\right)\Big|_0^{2\pi}$$

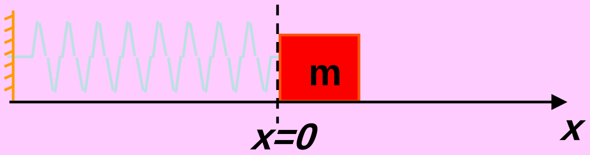
$$w = aF_o\sin\phi(\pi)$$

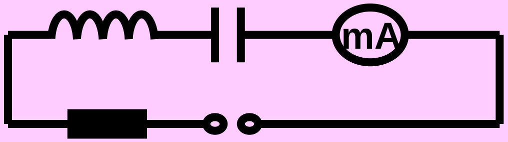
$$w = \pi aF_o\sin\phi$$

1.8: Oscillation with One Degree of Freedom;

System with one degree of freedom







Loaded spring

In the case of simple pendulum **LC circuit**

These os[redacted]lace about the mean position. All these systems have one degree of freedom.

For oscillatory with one degree of freedom, the displacement the "Moving particles" depends upon the SHM

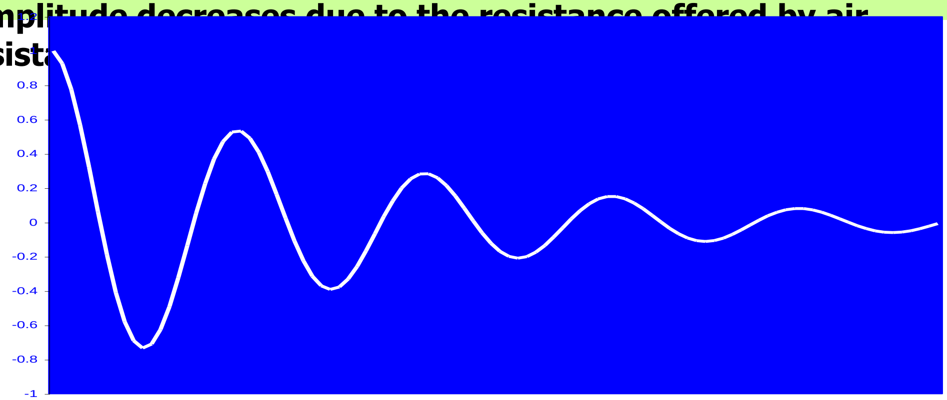
[redacted]

Damped Oscillations;

In actual particle the oscillatory system experiences fractional or resistive forces. Due to these reasons the oscillations get damped.

In the case of pendulum, the ampl

In the case of LC circuit, the resist



1.9: Linearly and Superposition Principles:

linear homogeneous equation.

non-linear.

non-homogeneous **non-linear**

$$M\frac{d^2y}{dt^2} = -Mg\sin\theta$$

$$M\frac{d^2y}{dt^2} = -Mg\,\theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\dots\dots$$

$$M\frac{d^2y}{dt^2} = -Mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\dots\dots\right)$$

Linear homogeneous equations:

* One of the important properties of linear homogeneous equations is that the sum of any two solutions is a solution by itself. This property is not true in the case of non-linear equation.

[redacted]

[redacted] be taken sufficiently near to Zero, then equation (3)

This equation is linear & homogeneous

[redacted]

(Differential displacement & velocity)

[redacted]

$X = y_1 + y_2$

The equations (7) and (8) are identical, only if

$$\frac{d^2}{dt^2}(y_1 + y_2) = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_2}{dt^2} ----(9)$$

$$-w^2(y_1 + y_2) = -w^2 y_1 - w^2 y_2 ------(10)$$

$$A({y_1}^2 + {y_2}^2) = A(y_1 + y_2)^2 ------(11)$$

$$B({y_1}^3 + {y_2}^3) = B(y_1 + y_2)^3 ------(12)$$

$$C({y_1}^4 + {y_2}^4) = C(y_1 + y_2)^4 ------(13)$$

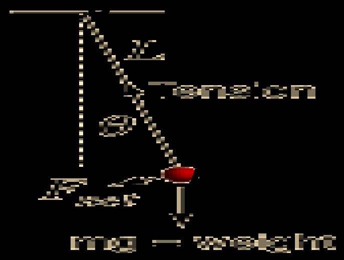
Equations (9) and (10) are true, But equations (11),(12) and (13) are true only, A=0,B=0, and C=0.

When A,B,C,….etc are Zero, the equation become linear. Hence superposition principle is true only in the case of homogeneous linear equation.

Also the sum of any two solutions is also a solution of the homogenous linear equation.

All harmonic oscillators given in equation (9) and (10) obey superposition principle.

1.11: Simple Pendulum:



Force along the string =Mg Cosθ

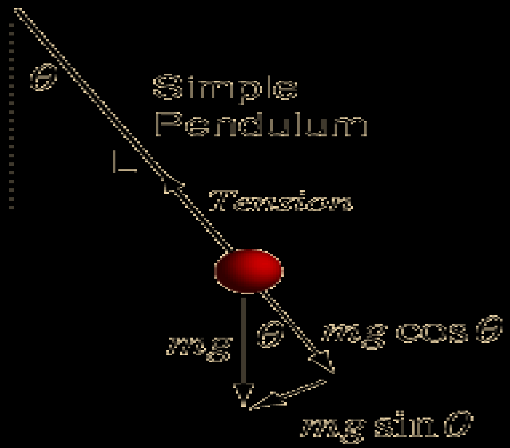
Force perpendicular to the string = Mg Sinθ

Mg Cosθ

Mg Cosθ = T

$$Sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ....$$

F= -Mg Sinθ



For small angular displacementθ;

Tangential force, F= Mgθ

$$\frac{dy}{dt} = l\frac{d\theta}{dt}$$

The linear displacement, y=ℓ θ →

$$\frac{d^2y}{dt^2} = l\frac{d^2\theta}{dt^2}$$

Acceleration,

$$F = ma' = Ml\frac{d^2\theta}{dt^2}$$

∴ Force,

$$Ml\frac{d^2\theta}{dt^2} = -Mg\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

nd law;

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 - - - - - - (1)$$

$$\frac{d^2y}{dt^2} + w^2\theta = 0 - - - - - - (2)$$

$$w^2 = \frac{g}{l}$$

n is similar to the $w = \sqrt{\frac{g}{l}}$ f SHM

$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$

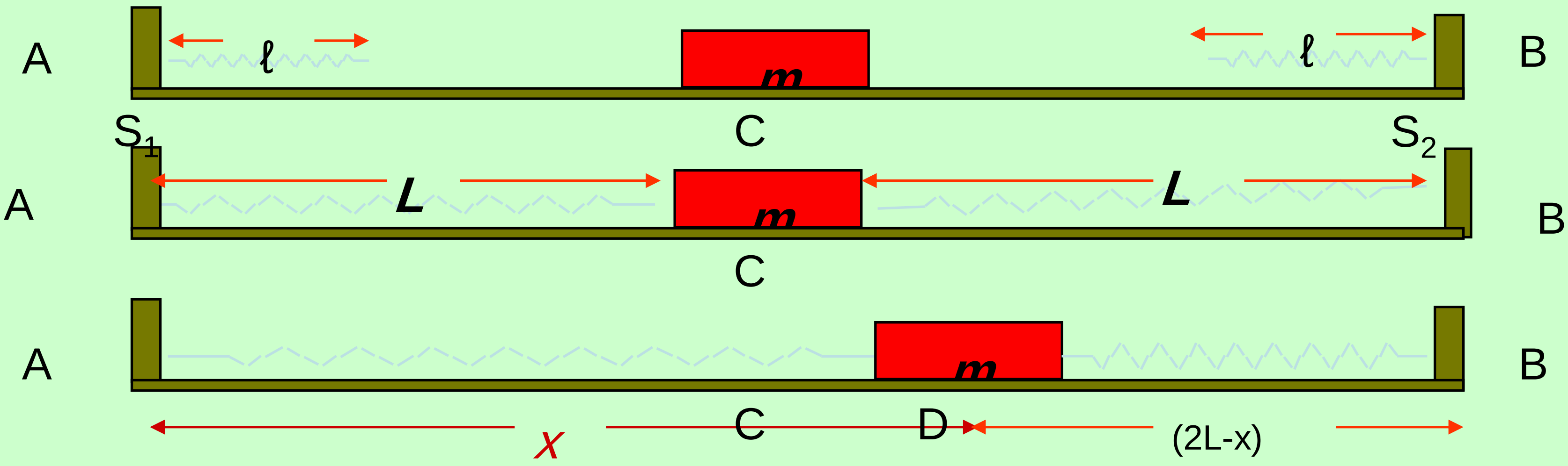
$$\therefore t = 2\pi\sqrt{\frac{l + \frac{2}{5}r^2/}{g}}$$

$$l + \frac{2}{5}r^2/$$

In case of a simple pendulum

If the size of the bob is large, a correction to be applied

1.12: Simple Harmonic Oscillation of a Mass between two springs:-



Here AC=BC=L

At C the mass is equally pulled by both the springs and it is the equilibrium position.

When the mass M is displaced from its equilibrium position and left, it excites SHO.

Let, at any instant, D be the displaced position of the mass M.

Here AD=x, and BD=(2L-x)

Let the tension per unit displacement in the spring be K.

The displacement of the spring of S_1 is (x-L) and it extents a force =K(x-L) in the direction DA.

*** The displacement of the spring of S_2 is (2L-x-ℓ) and it extents a force =K(2L-x-ℓ) in the direction DB.**

The resultant force on the mass M= K(2L-x-L)- K(x-L) in the direction DB

= 2Lk-xk-Lk-kx+kL

=2Lk-2xk=2k(L-x)

= -2k (x-L) in the direction DB

According to Newton's second law of motion;

$$F = M\frac{d^2x}{dt^2} = -2k(x-L) - - - - - - - - (1)$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{2k}{M}(x-L)$$

$$or \quad \frac{d^2x}{dt^2} + \frac{2k}{M}(x-L) = 0 - - - - - - - - (2)$$

Taking the displacement from the m... n;

x-L=y → x=y+L

$$\frac{dx}{dt} = \frac{dy}{dt} \rightarrow \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

Differentiating twice;

$$\frac{d^2y}{dt^2} + \frac{2k}{M}y = 0 - - - - - - - - (3)$$

Substituting these values in eq(2)

$$\frac{d^2y}{dt^2} + w^2y = 0 - - - - - - - - (4)$$

Th... ...milar to the equation ...

$$w^2 = \frac{2k}{M} \qquad w = \sqrt{\frac{2k}{M}}$$

From eqs(3) and (4);

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{M}{2k}} \quad - - - - - (5)$$

Consider two spring S1 and S2 each having a length l in the relaxed position.
Mass M is placed midway between the two springs on a frictionless surface.
One end of the spring S1 is attached to a rigid wall A and the other end is attached to the Mass M. Similarly one end of the spring S2 is attached to a rigid wall at B and the other end is connected to the mass M.